

Inflation from Tsunami-waves

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Abstract

We investigate inflation driven by the evolution of highly excited *quantum states* within the framework of out of equilibrium field dynamics. These states are characterized by a non-perturbatively large number of quanta in a band of momenta but with *vanishing* expectation value of the scalar field. They represent the situation in which initially a non-perturbatively large energy density is localized in a band of high energy quantum modes and are coined tsunami-waves. The self-consistent evolution of this quantum state and the scale factor is studied analytically and numerically. It is shown that the time evolution of these quantum states lead to two consecutive stages of inflation under conditions that are the quantum analogue of slow-roll. The evolution of the scale factor during the first stage has new features that are characteristic of the quantum state. During this initial stage the quantum fluctuations in the highly excited band build up an effective homogeneous condensate with a non-perturbatively large amplitude as a consequence of the large number of quanta. The second stage of inflation is similar to the usual classical chaotic scenario but driven by this effective condensate. The excited quantum modes are already superhorizon in the first stage and do not affect the power spectrum of scalar perturbations. Thus, this tsunami quantum state provides a field theoretical justification for chaotic scenarios driven by a classical homogeneous scalar field of large amplitude.

1 Introduction

A wealth of observational evidence from the temperature anisotropies in the cosmic microwave background favor inflation as the mechanism to produce the primordial density perturbations[1, 2]. Thus, inflationary cosmology emerges as the leading theoretical framework to explain not only the long-standing shortcomings of standard big bang cosmology but also to provide a testable paradigm for structure formation[3]-[6]. The recent explosion in the quantity and quality of data on temperature anisotropies elevates inflation to the realm of an experimentally testable scenario that leads to robust predictions that withstand detailed scrutiny[1, 2].

However at the level of implementation of an inflationary proposal, the situation is much less satisfactory. There are very many different models for inflation motivated by particle physics and

most if not all of them invoke one or several scalar fields, the inflaton(s), whose dynamical evolution in a scalar potential leads to an inflationary epoch[3]-[6]. The inflaton field is a scalar field that provides an effective description for the fields in the grand unified theories. Furthermore there is the tantalizing prospect of learning some aspects of the inflationary potential (at least the part of the potential associated with the last few e-folds) through the temperature anisotropies of the cosmic microwave background[7].

Most treatments of inflation study the evolution of the inflaton via the *classical* equations of motion in the scalar potential and the effect of quantum fluctuations is typically neglected in the dynamics of the inflaton. Furthermore, since inflation redshifts inhomogeneities very fast, the classical evolution is studied in terms of a *homogeneous classical scalar* field. The quantum field theory interpretation is that this classical, homogeneous field configuration is the expectation value of a quantum field operator in a translational invariant quantum state. While the evolution of this coherent field configuration (the expectation value or order parameter) is studied via classical equations of motion, quantum fluctuations of the scalar field around this expectation value are treated perturbatively and are interpreted as the seeds for scalar density perturbations of the metric[3]-[6].

A fairly broad catalog of inflationary models based on scalar field dynamics labels these either as ‘small field’ or ‘large field’[7]. In the ‘small field’ category the scalar field begins its evolution with an initial value very near the origin of the scalar potential and rolls down towards larger values, an example is new inflation[3, 4]. In the ‘large field’ category, the scalar field begins very high up in the potential hill and rolls down towards smaller values, an example is chaotic inflation[3, 4].

It is only recently that the influence of quantum fluctuations of the scalar fields in the dynamical evolution of matter and geometry has been studied self-consistently, mainly associated with the dynamics of non-equilibrium phase transitions[8, 9] in models that fall, broadly, in the ‘small field’ category. The conclusion of these studies is that a treatment of the quantum fluctuations that couple self-consistently to the dynamics of the metric provides a solid quantum field theoretical framework that justifies microscopically the picture based on classical inflation. At the same time these studies provide a deeper understanding of the quantum as well as classical aspects of inflation and inflationary perturbations. They clearly reveal the classicalization of initial quantum fluctuations[8, 9], and furnish a microscopic explanation (and derivation) of the effective, homogeneous classical inflaton[9].

The purpose of this article is to provide a quantum treatment of models whose classical counterpart are large field models. The *classical* description in these models begins with a homogeneous inflaton scalar with very large amplitude $\phi \sim M_{Pl}$ [3, 4, 6], i.e, very high up in the scalar potential well. The question of how is this initial condition achieved is typically answered in terms of a probabilistic distribution of initial conditions[3, 4], as is the case for chaotic inflation.

Instead, in this article we study the dynamics that results from the evolution of a *quantum state* which drives the dynamics of the scale factor through the expectation value of the energy momentum tensor.

The initial state the we consider is a squeezed state with a large number of particles distributed in a shell in momentum. While squeezed states had been studied in quantum optics[10] and also in cosmology[11] the state considered in this article is similar to that invoked to model the evolution of highly excited initial quantum states to describe the dynamics of heavy ion collisions. These states are characterized by a large population of quanta localized in momentum shells and had been coined tsunami-waves in[12, 13, 14]. The time evolution of these states leads to the formation of

a non-equilibrium plasma and features properties similar to those expected to occur in the cooling and expansion of a quark gluon plasma after the collision[15].

The goals of this article:

We here adapt the ideas and concepts in refs.[12, 13, 14] to study the self-consistent dynamics of the metric and the evolution of a highly excited quantum state with the goal of providing a quantum description of large field inflationary models *without assuming an expectation value for the scalar field*. We introduce novel quantum states, which are the cosmological counterpart of the tsunami-waves introduced in[12, 13, 14] with the following properties:

- *Pure states:* the states under consideration, defined by the wave-functional of the form (18) are *pure states*. Other proposals involving a mixed state density matrix as initial state are discussed in secs. III, IV and the Appendix.
- *Vanishing expectation value of the scalar field:* Unlike most models of classical chaotic inflation in which the scalar field obtains an expectation value, taken as a classical field, the expectation value of the scalar field in the tsunami-wave states given by the wave-functional (18) *vanishes*. [See sec. IV for a non-zero expectation value of the scalar field].
- *Highly excited initial modes:* the tsunami-wave state described by eq. (18) with the covariance kernel given by eqs. (23), (30) describes a state for which the modes *inside* a band are occupied with a non-perturbatively large $\mathcal{O}(1/\lambda)$ number of (adiabatic) quanta. This requires the use of non-perturbative methods as the large N approach and makes the states considered to be very far from the vacuum. We remark that very high energy modes, those that will become superhorizon during the last 10 or so e-folds, hence are of cosmological importance today, must be in the vacuum state so as not to lead to a large amplitude of scalar density perturbations[3, 6, 8].

This type of *quantum states* is clearly a novel concept, it presents an alternative to typical inflationary scenarios that invoke the dynamics of a *classical* scalar field which in most cases ignore the quantum dynamics.

The quantum nature of the tsunami states is due to the coherence between different modes and gives rise to dynamical consequences. (See [14]). In the present case the system is effectively classical only after the redshift has assembled the modes into a zero-mode effective condensate.

The tsunami-wave states described above are the simplest states and will be the focus of our study. These states can be generalized to describe mixed-state density matrices and to also allow for an expectation value of the scalar field. These generalizations are described in sections III.C and IV and in the appendix and are found to lead qualitatively the same features revealed by the simpler pure states.

We establish the conditions under which such quantum state leads to inflationary dynamics and study in detail the self-consistent evolution of this quantum state and the space-time metric.

We emphasize that we are *not* proposing here yet a new model of inflation. Instead we focus on inflation driven by the evolution of a *quantum* state, within the framework of familiar models based on scalar fields with typical quartic potentials. This is in contrast with the usual approach in which the dynamics is driven by the evolution of a homogeneous *classical* field of large amplitude.

Brief summary: We find that inflation occurs under fairly general conditions that are the *quantum* equivalent of slow-roll. There are *two* consecutive but distinct inflationary stages: the

first one is completely determined by the quantum features of the state. Even when the expectation value of the scalar field *vanishes at all times* in this quantum state, the dynamics of the first stage gives rise to the emergence of an *effective classical homogeneous condensate*. The amplitude of the effective condensate is non-perturbatively large as a consequence of the non-perturbatively large number of quanta in the band of excited wavevectors. The second stage is similar to the familiar classical chaotic scenario, and can be interpreted as being driven by the dynamics of the effective homogeneous condensate. The band of excited quantum modes, if not superhorizon initially they cross the horizon during the first stage of inflation, hence they do not modify the power spectrum of scalar density perturbations on wavelengths that are of cosmological relevance today. Actually, in the explicit examples worked out here, the excited modes are initially superhorizon due to the generalized slow-roll condition. Therefore, in a very well defined manner, tsunami quantum states provide a quantum field theoretical justification, a microscopic basis, for chaotic inflation, explaining the classical dynamics of the homogeneous scalar field.

In section II we introduce the quantum state, obtain the renormalized equations of motion for the self-consistent evolution of the quantum state and the scale factor. In section III we provide detailed analytic and numerical studies of the evolution and highlight the different inflationary stages. In section IV we discuss generalized scenarios. The summary of results is presented in the conclusions. We derive the equations of motion for mixed states in the appendix.

2 Initial state and equations of motion

As emphasized in the introduction, while most works on inflation treat the dynamics of the inflaton field at the classical level, we use a quantum description of the inflaton.

We focus on the possibility of inflation through the dynamical *quantum* evolution of a highly excited initial state with large energy density. Consistently with inflation at a scale well below the Planck energy, we treat the inflaton field describing the matter as a quantum field whereas gravity is treated semiclassically.

The dynamics of the classical space-time metric is determined by the Einstein equations with a source term given by the expectation value of the energy momentum tensor of the quantum inflaton field. The quantum field evolution is calculated in the resulting metric.

Hence we solve *self-consistently* the coupled evolution equations for the classical metric and the quantum inflaton field.

We assume that the universe is homogeneous, isotropic and spatially flat, thus it is described by the metric,

$$ds^2 = dt^2 - a^2(t) d\vec{x}^2. \quad (1)$$

Anticipating the need for a non-perturbative treatment of the evolution of the quantum state, we consider an inflaton model with an N -component scalar inflaton field $\vec{\Phi}(x)$ with quartic self-coupling. We then invoke the large N limit as a non-perturbative tool to study the dynamics[8, 9, 13, 12, 16]. This choice is not only motivated by the necessity of a consistent non-perturbative treatment but also because any grand unified field theory will contain a large number of scalar fields, thus justifying a large N limit on more physical grounds.

The matter action and Lagrangian density are given by

$$S[\vec{\Phi}] = \int d^4x \mathcal{L}_m = \int d^4x a^3(t) \left[\frac{1}{2} \dot{\vec{\Phi}}^2(x) - \frac{1}{2} \frac{(\vec{\nabla} \vec{\Phi}(x))^2}{a^2(t)} - V(\vec{\Phi}(x)) \right], \quad (2)$$

$$V(\vec{\Phi}) = \frac{m^2}{2} \vec{\Phi}^2 + \frac{\lambda}{8N} (\vec{\Phi}^2)^2 + \frac{1}{2} \xi \mathcal{R} \vec{\Phi}^2, \quad (3)$$

and we will consider $m^2 > 0$, postponing the discussion of the case $m^2 < 0$ to subsequent work. Here $\mathcal{R}(t)$ stands for the scalar curvature

$$\mathcal{R}(t) = 6 \left(\frac{\ddot{a}(t)}{a(t)} + \frac{\dot{a}^2(t)}{a^2(t)} \right), \quad (4)$$

The ξ -coupling of $\vec{\Phi}^2(x)$ to the scalar curvature $\mathcal{R}(t)$ has been included in the Lagrangian since it is necessary for the renormalizability of the theory.

The discussion of the alternative inflationary mechanism that we are proposing and the physical description of the quantum states becomes more clear in conformal time

$$\mathcal{T} = \int^t \frac{dt'}{a(t')} \quad (5)$$

in terms of which the metric is conformal to that in Minkowski space-time

$$ds^2 = a^2(\mathcal{T}) (d\mathcal{T}^2 - d\mathbf{x}^2). \quad (6)$$

We introduce the conformally rescaled field

$$\vec{\Psi}(\mathcal{T}, \mathbf{x}) = a(t) \vec{\Phi}(t, \mathbf{x}) \quad (7)$$

in terms of which the matter action becomes

$$\mathcal{S}[\vec{\Psi}] = \int d\mathcal{T} d^3x \left\{ \frac{1}{2} [(\partial_{\mathcal{T}} \vec{\Psi})^2 - (\nabla \vec{\Psi})^2] - a^4(\mathcal{T}) V \left[\frac{\vec{\Psi}}{a(\mathcal{T})} \right] + a^2(\mathcal{T}) \frac{\mathcal{R}}{12} \vec{\Psi}^2 \right\}. \quad (8)$$

Since we are interested in describing the time evolution of an initial quantum state, we pass on to the Hamiltonian description in the Schrödinger representation. This procedure begins by obtaining the canonical momentum conjugate to the quantum field, $\vec{\Pi}(\mathcal{T}, \mathbf{x})$, and the Hamiltonian density $\mathcal{H}(\mathcal{T}, \mathbf{x})$

$$\begin{aligned} \vec{\Pi}(\mathcal{T}, \mathbf{x}) &= \vec{\Psi}'(\mathcal{T}, \mathbf{x}), \\ \mathcal{H}(\mathcal{T}, \mathbf{x}) &= \frac{1}{2} \vec{\Pi}^2 + \frac{1}{2} (\nabla \vec{\Psi})^2 + a^4(\mathcal{T}) V \left[\frac{\vec{\Psi}}{a(\mathcal{T})} \right] - a^2(\mathcal{T}) \frac{\mathcal{R}}{12} \vec{\Psi}^2, \\ H(\mathcal{T}) &= \int d^3\mathbf{x} \mathcal{H}(\mathcal{T}, \mathbf{x}), \end{aligned} \quad (9)$$

where the prime denotes derivative with respect to the conformal time \mathcal{T} .

In the Schrödinger representation the canonical momentum is given by

$$\Pi^a(\mathcal{T}, \mathbf{x}) = -i \frac{\delta}{\delta \Psi^a(\mathcal{T}, \mathbf{x})} \quad ; \quad a = 1, \dots, N. \quad (10)$$

The time evolution of the wave-functional $\Upsilon[\vec{\Psi}; \mathcal{T}]$ is obtained from the functional Schrödinger equation

$$i \frac{\partial}{\partial \mathcal{T}} \Upsilon[\vec{\Psi}; \mathcal{T}] = H \left[\frac{\partial}{\partial \vec{\Psi}}; \vec{\Psi} \right] \Upsilon[\vec{\Psi}; \mathcal{T}] \quad (11)$$

The implementation of the large N limit begins by writing the field as follows

$$\begin{aligned}\vec{\Psi}(\mathbf{x}, T) &= (\sigma(\mathbf{x}, T), \vec{\pi}(\mathbf{x}, T)) \\ &= (\sqrt{N}\psi(T) + \chi(\mathbf{x}, T), \vec{\pi}(\mathbf{x}, T)),\end{aligned}\tag{12}$$

where we choose the ‘1’-axis in the direction of the expectation value of the field and we collectively denote by $\vec{\pi}$ the $N - 1$ perpendicular directions.

$$\begin{aligned}\psi(T) &= \langle \sigma(\mathbf{x}, T) \rangle \\ \langle \vec{\pi}(\mathbf{x}, T) \rangle &= \langle \chi(\mathbf{x}, T) \rangle = 0,\end{aligned}\tag{13}$$

where the expectations value above are obtained in the state represented by the wave-functional $\Upsilon[\vec{\Psi}; T]$ introduced above.

The leading order in the large N limit can be efficiently obtained by functional methods (see refs.[8, 9, 13, 14, 16] and references therein). The contributions of χ to the equations of motion are subleading (of order $1/N$) in the large N limit[8, 16].

It is convenient to introduce the spatial Fourier modes of the quantum field

$$\vec{\pi}_k(T) = \int d^3x \vec{\pi}(\mathbf{x}, T) e^{i\mathbf{k}\cdot\mathbf{x}}\tag{14}$$

In leading order in the large N limit, the explicit form of the Hamiltonian is given by[8, 9, 13, 14, 16]

$$\begin{aligned}H(T) &= N\mathcal{V}h_{cl}(T) - \frac{\lambda}{8N} \left(\sum_k \langle \vec{\pi}_k \cdot \vec{\pi}_{-k} \rangle \right)^2 + \sum_k H_k(T), \\ h_{cl}(T) &= \frac{1}{2} \psi'^2(T) + \frac{a^2(T)}{2} m^2 \psi^2(T) + \frac{\lambda}{8} \psi^4(T), \\ H_k(T) &\equiv -\frac{1}{2} \frac{\delta^2}{\delta \vec{\pi}_k \cdot \delta \vec{\pi}_{-k}} + \frac{1}{2} \omega_k^2(T) \vec{\pi}_k \cdot \vec{\pi}_{-k}\end{aligned}\tag{15}$$

$$\omega_k^2(T) \equiv k^2 + a^2(T) \left[\mathcal{M}^2(T) - \frac{\mathcal{R}(T)}{6} \right],\tag{16}$$

$$\mathcal{M}^2(T) \equiv m^2 + \xi \mathcal{R} + \frac{\lambda}{2} \frac{\psi^2}{a^2(T)} + \frac{\lambda}{2} \frac{\langle \vec{\pi}^2 \rangle}{N a^2(T)}.\tag{17}$$

where \mathcal{V} is the comoving volume. We assume spherically symmetric distributions in momentum space.

That is, in the large N limit the Hamiltonian operator (9) becomes a time dependent c-number contribution plus a quantum mechanical contribution, $\sum_k H_k(T)$, given by a collection of harmonic oscillators with time-dependent frequencies, coupled only through the quantum fluctuations $\langle \vec{\pi}_k \cdot \vec{\pi}_{-k} \rangle$.

In eqs.(15)-(17) the scale factor $a(T)$ is determined self-consistently by the Einstein-Friedmann equations.

2.1 Tsunami initial states

To highlight the main aspects of the inflationary scenario proposed here, and to establish a clear difference with the conventional models, we now focus our discussion on the case of vanishing expectation value, i.e. $\psi(\mathcal{T}) = 0$, and a *pure quantum state*. The most general cases with mixed states described by density matrices and non-vanishing expectation value of the field are discussed in detail in sections IIIC and IV and in the appendix.

For $\psi(\mathcal{T}) = 0$ the quantum Hamiltonian (15) becomes a sum of harmonic oscillators with time dependent frequencies that depend on the quantum fluctuations. Therefore, we propose a Gaussian wave-functional of the form

$$\Upsilon \left[\vec{\Psi}; \mathcal{T} \right] = \mathcal{N}_\Upsilon(\mathcal{T}) \prod_k e^{-\frac{A_k(\mathcal{T})}{2} \vec{\pi}_k \cdot \vec{\pi}_{-k}} . \quad (18)$$

The functional Schrödinger equation (11) in this case leads to evolution equations for the normalization factor $\mathcal{N}_\Upsilon(\mathcal{T})$ and the covariance kernel $A_k(\mathcal{T})$ whose general form is found in the appendix (see also [8]). The evolution of the normalization factor is determined by that of A_k , while the equation for A_k is

$$iA'_k(\mathcal{T}) = A_k^2 - \omega_k^2(\mathcal{T}) , \quad (19)$$

where primes refer to derivatives with respect to conformal time. As described in the appendix for the general case, the above equation can be linearized by defining (see appendix)

$$A_k(\mathcal{T}) \equiv -i \frac{\varphi_k'^*(\mathcal{T})}{\varphi_k^*(\mathcal{T})} , \quad (20)$$

where the mode functions φ_k satisfy the equation

$$\varphi_k'' + \omega_k^2(\mathcal{T}) \varphi_k = 0 \quad (21)$$

In terms of these mode functions the self-consistent expectation value

$$\begin{aligned} \frac{\langle \vec{\pi}^2 \rangle}{N} &= \frac{1}{N} \int \frac{d^3 k}{(2\pi)^3} \langle \vec{\pi}_k \cdot \vec{\pi}_{-k} \rangle \\ \langle \vec{\pi}_k \cdot \vec{\pi}_{-k} \rangle &= \frac{N}{2A_{R,k}} = \frac{N}{2} |\varphi_k|^2 . \end{aligned} \quad (22)$$

We now must provide initial conditions on the wave functional to completely specify the dynamics. Choosing the initial (conformal) time at $\mathcal{T} = 0$ with $a(\mathcal{T} = 0) = 1$, the initial state is completely specified by furnishing the real and imaginary parts of the covariance A_k at the initial time. We parameterize these as¹

$$A_{R,k}(0) = \Omega_k \quad ; \quad A_{I,k}(0) = \omega_k(0) \delta_k \quad (23)$$

Choosing the Wronskian of the mode functions $\varphi_k(\mathcal{T})$ and its complex conjugate to be

$$\varphi_k \varphi_k'^* - \varphi_k' \varphi_k^* = 2i \quad (24)$$

¹In the case for which $\omega_k^2(0) < 0$ we choose $\omega_k(0) = \sqrt{k^2 + |\mathcal{M}^2(0) - \mathcal{R}(0)/6|}$.

determines the following initial condition on the mode functions (see appendix)

$$\varphi_k(0) = \frac{1}{\sqrt{\Omega_k}} \quad ; \quad \varphi'_k(0) = -[\omega_k(0)\delta_k + i\Omega_k]\varphi_k(0) . \quad (25)$$

where we also used eqs.(20) and (23).

An important alternative interpretation of these mode functions is that they form a basis for expanding the Heisenberg field operators (solution of the Heisenberg equations of motion)

$$\vec{\pi}(\vec{x}, T) = \int \frac{d^3k}{(2\pi)^3} \left[\vec{a}_k \varphi_k(T) e^{i\vec{k}\cdot\vec{x}} + \vec{a}_k^\dagger \varphi_k^*(T) e^{-i\vec{k}\cdot\vec{x}} \right] , \quad (26)$$

with $\vec{a}_k; \vec{a}_k^\dagger$ annihilation and creation operators, respectively, with canonical commutation relations. The Wronskian condition (24) ensures that the $\vec{\pi}(\vec{x}, T)$ fields and their conjugate momenta obey the canonical commutation relations at equal conformal times.

The physical interpretation of these initial states is highlighted by focusing on the occupation number of adiabatic states as well as on the probability distribution of field configurations.

- *Occupation number:* it is at this point where the description in terms of conformal time proves to be valuable. In conformal time the Hamiltonian in the large N limit is that of a collection of harmonic oscillators with time dependent frequencies. It is then convenient to introduce the adiabatic occupation number operator

$$\hat{n}_k(T) = \frac{1}{N} \left[\frac{H_k(T)}{\omega_k(T)} - \frac{1}{2} \right] , \quad (27)$$

with H_k given by eq. (15).

In particular the occupation number at the initial time is given by (see appendix)

$$n_k \equiv \langle \hat{n}_k(0) \rangle = \frac{[\omega_k(0) - \Omega_k]^2 + \omega_k^2(0) \delta_k^2}{4 \omega_k(0) \Omega_k} . \quad (28)$$

Here, the special case with $\Omega_k = \omega_k(0)$ and $\delta_k = 0$ corresponds to the adiabatic vacuum ($n_k = 0$). Instead, we study an initial state in which a band of wave-vectors are *populated with a non-perturbatively large number of particles*. More precisely, we consider initial states for which

$$\begin{aligned} n_k &= \mathcal{O}\left(\frac{1}{\lambda}\right) \quad \text{inside the excited band,} \\ n_k &= 0 \quad \text{outside the excited band.} \end{aligned} \quad (29)$$

where λ is the quartic self-coupling.

This is accomplished by choosing,

$$\begin{aligned} \frac{1}{\Omega_k} &= \mathcal{O}\left(\frac{1}{\lambda \omega_k(0)}\right) \quad \text{inside the excited band,} \\ \frac{1}{\Omega_k} &= \frac{1}{\omega_k(0)} \quad \text{and} \quad \delta_k = 0 \quad \text{outside the excited band.} \end{aligned} \quad (30)$$

These initial states are highly excited, the expectation value of the energy momentum tensor in these states leads to an energy density $\sim 1/\lambda$ and are, therefore, non-perturbative. We will refer to the case where the excited band is narrow as the narrow tsunami.

It must be stressed that the particle distribution n_k alone *partially* determines the initial state. As we see from eq.(25), the initial state is completely defined specifying *two* functions of k : Ω_k and δ_k .

- *Probability distribution*: an alternative interpretation of these initial states is obtained by focusing on the probability distribution of field configurations at the initial time. It is given by

$$\mathcal{P}[\vec{\pi}] = \left| \Upsilon \left[\vec{\Psi}; \mathcal{T} = 0 \right] \right|^2 = \mathcal{N}(0) \prod_k e^{-\Omega_k \vec{\pi}_k \cdot \vec{\pi}_{-k}} \quad (31)$$

An intuitive quantum mechanical picture of the wave-functional for the modes in the excited band is the following. At the initial time the instantaneous Hamiltonian corresponds to a set of harmonic oscillators of frequencies $\omega_k(0)$, while the width in field space of the initial Gaussian state is determined by $\Omega_k^{-1/2}$. For a mode in the vacuum state $1/\sqrt{\Omega_k} \sim 1/\sqrt{\omega_k(0)}$ and the typical amplitudes of the field are $\vec{\pi}_k \sim 1/\sqrt{\omega_k(0)}$ which is the typical width of the potential well. While for a mode inside the excited band the width in field space is $1/\sqrt{\Omega_k} \sim 1/\sqrt{\lambda \omega_k(0)}$ [see eq.(30)]. Thus, large amplitude field configurations with $\vec{\pi}_{k \approx k_0} \sim 1/\sqrt{\lambda \omega_k(0)} \gg 1/\sqrt{\omega_k(0)}$ have a probability of $\mathcal{O}(1)$, i.e, large amplitude configurations within the band of excited wave-vectors are *not* suppressed. That is, the width of the probability distribution for these modes is much larger than the typical size of the potential well and there is a non-negligible probability for finding field configurations with large amplitudes of $\mathcal{O}(1/\lambda)$.

These highly excited initial states had been previously proposed as models to describe the initial stages of ultrarelativistic heavy ion collisions and had been coined ‘tsunami waves’[12, 13, 14]. They represent spherical shells (in momentum space) with large occupation numbers of quanta, describing a state with a large energy density with particles of a given momentum.

2.2 Back to comoving time: renormalized equations of motion

Having set up the initial value problem in terms of the tsunami-wave initial wave-functionals, the dynamics is now completely determined by the set of mode equations eqs.(21) with eqs.(16)-(17) and the initial conditions eqs.(25). However, in order to establish contact with more familiar results in the literature, it is convenient to re-write the equations of motion in comoving time. This is accomplished by the field rescaling given by eq.(7) which at the level of mode function results in introducing the comoving time mode functions $f_k(t)$ related to the conformal time ones $\varphi_k(\mathcal{T})$ as

$$f_k(t) = \frac{\varphi_k(\mathcal{T})}{a(t)} \quad (32)$$

The equations of motion in comoving time for these mode functions are

$$\ddot{f}_k(t) + 3H(t)\dot{f}_k(t) + \left[\frac{k^2}{a^2(t)} + \mathcal{M}^2(t) \right] f_k(t) = 0 \quad (33)$$

$$\mathcal{M}^2(t) = m^2 + \xi \mathcal{R}(t) + \frac{\lambda}{4} \int \frac{d^3 k}{(2\pi)^3} |f_k(t)|^2 \quad (34)$$

$$f_k(0) = \frac{1}{\sqrt{\Omega_k}} ; \quad \dot{f}_k(0) = -[\omega_k(0) \delta_k + H(0) + i\Omega_k] f_k(0) . \quad (35)$$

The Einstein-Friedmann equation are,

$$H^2(t) = \left(\frac{\dot{a}(t)}{a(t)} \right)^2 = \frac{8\pi\rho_0}{3M_{Pl}^2} , \quad \rho_0 = \langle T_{00} \rangle , \quad (36)$$

where the expectation value is taken in the time evolved quantum state. It is straightforward to see that the expectation value of the energy-momentum tensor has the perfect fluid form, as a consequence of the homogeneity and isotropy of the system[8, 9].

Thus the set of equations (33)-(36) provide a closed set of self-consistent equation for the dynamics of the quantum state and the space-time metric.

2.2.1 Renormalized Equations of Motion in the Large N limit

The set of equations that determine the dynamics of the quantum state *and* the scale factor need to be renormalized. The field quantum fluctuations

$$\frac{\langle \vec{\pi}^2 \rangle}{N} = \int \frac{d^3 k}{2(2\pi)^3} |f_k(t)|^2 ,$$

requires subtractions which are absorbed in a renormalization of the mass, coupling to the Ricci scalar and coupling constant. The expectation value of the stress tensor also requires subtractions (but not multiplicative renormalization). Since the divergence structure is determined by the large energy, short distance behavior, the band of excited modes does not influence the renormalization aspects. Therefore, we use the extensive work on the renormalization program which is available in the literature referring the reader to references[8, 9] for details. We here summarize the aspects that are most relevant for the present discussion.

First, it is convenient to introduce the following dimensionless quantities,

$$\begin{aligned} \tau = m t \quad ; \quad h(\tau) = \frac{H(t)}{m} \quad ; \quad q = \frac{k}{m} \quad ; \\ \omega_q = \frac{\omega_k}{m} \quad ; \quad \Omega_q = \frac{\Omega_k}{m} \quad ; \quad g = \frac{\lambda}{8\pi^2} \quad ; \quad f_q(\tau) = \sqrt{m} f_k(t) , \end{aligned} \quad (37)$$

where m and λ stand for the renormalized mass of the inflaton and the renormalized self-coupling, respectively[8]. In terms of these dimensionless quantities we now introduce the dimensionless and fully renormalized expectation value of the self-consistent field as

$$\begin{aligned} g\Sigma(\tau) &\equiv \frac{\lambda}{2m^2} \langle \pi^2(t) \rangle_R \\ \Sigma(\tau) &= \int_0^\infty q^2 dq \left[|f_q(\tau)|^2 - \frac{1}{q a(\tau)^2} + \frac{\Theta(q-1)}{2q^3} \left(\frac{\mathcal{M}^2(\tau)}{m^2} - \frac{\mathcal{R}(\tau)}{6m^2} \right) \right] . \end{aligned} \quad (38)$$

where the terms subtracted inside the integrand renormalize the mass, coupling to the Ricci scalar and the coupling constant[8, 9]. The dimensionless and renormalized expressions for the energy

density ϵ and pressure p are given by

$$\begin{aligned}
\epsilon(\tau) &\equiv \frac{\lambda}{2N m^4} \langle T^{00} \rangle_R = \\
&= \frac{g\Sigma(\tau)}{2} + \frac{[g\Sigma(\tau)]^2}{4} + \frac{g}{2} \int q^2 dq \left\{ |\dot{f}_q(\tau)|^2 - S_1(q, \tau) + \frac{q^2}{a^2(\tau)} [|f_q(\tau)|^2 - S_2(q, \tau)] \right\} \\
p(\tau) &\equiv \frac{\lambda}{2N m^4} \langle T^{ii} \rangle_R \\
(p + \epsilon)(\tau) &= g \int q^2 dq \left\{ |\dot{f}_q(\tau)|^2 - S_1(q, \tau) + \frac{q^2}{3a^2(\tau)} [|f_q(\tau)|^2 - S_2(q, \tau)] \right\} .
\end{aligned} \tag{39}$$

Where the renormalization subtractions S_1 and S_2 are given by, [8, 9]

$$\begin{aligned}
S_1(q, \tau) &= \frac{q}{a^4(\tau)} + \frac{1}{2qa^4(\tau)} [B(\tau) + 2\dot{a}^2] \\
&+ \frac{\Theta(q-1)}{8q^3 a^4(\tau)} [-B(\tau)^2 - a(\tau)^2 \ddot{B}(\tau) + 3a(\tau)\dot{a}(\tau)\dot{B}(\tau) - 4\dot{a}^2(\tau)B(\tau)] , \\
S_2(q, \tau) &= \frac{1}{qa^2(\tau)} - \frac{1}{2q^3 a^2(\tau)} B(\tau) + \frac{\Theta(q-1)}{8q^5 a^2(\tau)} \left\{ 3B(\tau)^2 + a(\tau) \frac{d}{d\tau} [a(\tau)\dot{B}(\tau)] \right\} , \\
B(\tau) &\equiv a^2(\tau) [1 + g\Sigma(\tau)] .
\end{aligned} \tag{40}$$

We choose here $\xi = 0$ (minimal coupling), the renormalization point $\kappa = |m|$ and $a(0) = 1$.

In summary, the set of coupled, self-consistent equations of motion for the quantum state and the scale factor are

$$\left[\frac{d^2}{d\tau^2} + 3h(\tau) \frac{d}{d\tau} + \frac{q^2}{a^2(\tau)} + 1 + g\Sigma(\tau) \right] f_q(\tau) = 0 \tag{41}$$

$$f_q(0) = \frac{1}{\sqrt{\Omega_q}} \quad ; \quad \dot{f}_q(0) = -[\omega_q \delta_q + h(0) + i\Omega_q] f_q(0) \tag{42}$$

$$\omega_q = \sqrt{q^2 + \left| 1 + g\Sigma(0) - \frac{\mathcal{R}(0)}{6m^2} \right|} , \tag{43}$$

plus the Einstein-Friedmann equation of motion for the scale factor

$$h^2(\tau) = L^2 \epsilon(\tau) \quad , \quad \text{where } L^2 \equiv \frac{16 \pi N m^2}{3 M_{Pl}^2 \lambda} . \tag{44}$$

with $g\Sigma(\tau)$ and $\epsilon(\tau)$ given by eqs. (38) and (39) respectively.

In order to implement the numerical analysis of the set of eqs. (41)-(42), (38) and (44) we introduce an ultraviolet momentum cutoff Λ . For the cases considered in this article we choose $\Lambda \sim 200$ and found almost no dependence on the cutoff for larger values. As befits a scalar inflationary model, the scalar self-coupling is constrained by the amplitude of scalar density perturbations to be $\lambda \sim 10^{-12}$ [3, 5] implying that $g < 10^{-13}$. Therefore, the subtractions can be neglected because $S_i \sim O(g\Lambda^4) < 10^{-4}$.

The initial state is defined by specifying the Ω_q and δ_q . We determine the range of these parameters Ω_q and δ_q by the excitation spectrum for the tsunami-wave initial state, as well as the condition that lead to inflationary stage. This will be studied in detail in the next section.

3 Tsunami inflation

As emphasized in the previous section, the scenario under consideration is very different from the popular treatments of inflation based on the evolution of *classical* scalar inflaton field[2, 3, 5, 6]. In these scenarios all of the initial energy is assumed to be in a zero mode (or order parameter) at the beginning of inflation and the quantum fluctuations are taken to be perturbatively small with a negligible contribution to the energy density and the evolution of the scale factor.

In contrast to this description, our proposal highlights the dynamics of the *quantum states* as the driving mechanism for inflation. The initial quantum states under consideration correspond to a band of quantum modes in highly excited states, thus the name ‘tsunami-wave’[12, 13, 14]. This initial state models a cosmological initial condition in which the energy density is non-perturbatively large, but concentrated in the quanta rather than in a zero mode.

We now study under which general conditions such a state can lead to a period of inflation that satisfies the cosmological constraints for solving the horizon and entropy problems entailing the necessity for about 60 e-folds of inflation.

It is understood that inflation takes place whenever the expansion of the universe accelerates, i.e,

$$\frac{\ddot{a}}{a} = h^2 + \dot{h} = -\frac{L^2}{2}[\epsilon + 3p] > 0 , \quad (45)$$

with L given in eq. (44) and ϵ and p given by eqs. (39).

While our full analysis rely on the numerical integration of the above set of equations, much we learn by considering the *narrow tsunami case*.

3.1 Analytical study: the narrow tsunami case

Before proceeding to a full numerical study of the equations of motion, we want to obtain an analytic estimate of the conditions under which a tsunami initial quantum state would lead to inflation.

Our main criterion for such initial state to represent high energy excitations is that the number of quanta in the band of excited modes is of $\mathcal{O}(1/\lambda)$. This criterion, as explained above, is tantamount to requiring that field configurations with non-perturbative amplitudes have non-negligible functional probability. Progress can be made analytically by focusing on the case in which the band of excited field modes is *narrow* i.e, its width Δk is such that $\Delta k \ll k_0$ or in terms of dimensionless quantities $\Delta q/q_0 \ll 1$. We introduce the following smooth distribution

$$\Omega_q = \frac{\omega_q}{1 + \frac{\mathcal{N}_\Omega}{g} e^{-\left[\frac{q-q_0}{\sqrt{2}\Delta q}\right]^2}} , \quad \text{with} \quad \frac{\Delta q}{q_0} \ll 1 , \quad (46)$$

with ω_q given by eq. (43) and \mathcal{N}_Ω a normalization constant that fixes the value of the total energy.

In addition, we choose $\delta_q = -h(0)/\omega_q$ as we discuss below in eq.(57).

This initial distribution posses the main features of the tsunami state described in the previous section. Since $g \ll 1$, we have for $q \sim q_0$,

$$\frac{1}{\Omega_q} \sim \frac{1}{g} \gg 1 \quad \Rightarrow \quad n_q \sim \frac{1}{g} \gg 1 \quad (47)$$

corresponding to highly excited states. While for $|q - q_0| \gg \Delta q$

$$\frac{1}{\Omega_q} \sim \frac{1}{\omega_q} \Rightarrow n_q \sim 0 . \quad (48)$$

Thus, these modes are in a quantum state near the conformal (adiabatic) vacuum at the initial time, with n_q the number of quanta defined by eq. (28) in terms of dimensionless variables. For these distributions (narrow tsunamis), the integral over mode functions for the quantum fluctuations $g\Sigma(\tau)$ [given by eq. (38)] is dominated by the narrow band of excited states with mode amplitudes $\sim 1/\sqrt{g}$ and can be approximated by

$$g\Sigma(\tau) = g \Delta q q_0^2 |f_{q_0}(\tau)|^2 + O(g) + O(g \Delta q) \simeq |\phi_{q_0}(\tau)|^2 , \quad (49)$$

where we have introduced the effective q_0 mode

$$\phi_{q_0}(\tau) \equiv \sqrt{g \Delta q} q_0 f_{q_0}(\tau) . \quad (50)$$

we note that the initial condition (42) and the tsunami-wave condition (47) entail that despite the presence of the coupling constant in its definition, the amplitude of the effective q_0 mode is of $\mathcal{O}(1)$.

The equation of motion for the effective q_0 -mode takes the form

$$\ddot{\phi}_{q_0}(\tau) + 3 h(\tau) \dot{\phi}_{q_0}(\tau) + \left[\frac{q_0^2}{a^2(\tau)} + 1 + |\phi_{q_0}(\tau)|^2 \right] \phi_{q_0}(\tau) = 0 . \quad (51)$$

The scale factor follows from

$$h^2(\tau) = L^2 \epsilon(\tau) , \quad (52)$$

with energy and pressure,

$$\begin{aligned} \epsilon(\tau) &= \frac{1}{2} |\dot{\phi}_{q_0}(\tau)|^2 + \frac{1}{2} |\phi_{q_0}(\tau)|^2 + \frac{1}{4} |\phi_{q_0}(\tau)|^4 + \frac{q_0^2}{2 a^2(\tau)} |\phi_{q_0}(\tau)|^2 , \\ (p + \epsilon)(\tau) &= |\dot{\phi}_{q_0}(\tau)|^2 + \frac{q_0^2}{3 a^2(\tau)} |\phi_{q_0}(\tau)|^2 . \end{aligned} \quad (53)$$

where we have neglected terms of $\mathcal{O}(g)$.

We will refer to the set of evolution equations (51)-(53) as the one mode approximation evolution equations.

In particular, within this one-mode approximation, the acceleration of the scale factor obeys

$$\frac{\ddot{a}(\tau)}{a(\tau)} = -L^2 \left[|\dot{\phi}_{q_0}(\tau)|^2 - \frac{|\phi_{q_0}(\tau)|^2}{2} - \frac{|\phi_{q_0}(\tau)|^4}{4} \right] \quad (54)$$

Therefore, the condition for an inflationary epoch, $\ddot{a} > 0$, becomes

$$|\dot{\phi}_{q_0}(\tau)|^2 < \frac{1}{2} |\phi_{q_0}(\tau)|^2 + \frac{1}{4} |\phi_{q_0}(\tau)|^4 . \quad (55)$$

A sufficient criterion that guarantees inflation is the *tsunami slow roll condition*

$$|\dot{\phi}_{q_0}(\tau)| \ll |\phi_{q_0}(\tau)| \quad (56)$$

The initial conditions (42) and the condition that $\Omega_{q_0} \sim g \ll 1$ imply that the tsunami slow roll condition (56) at early times is guaranteed if δ_{q_0} is such that

$$|\omega_{q_0} \delta_{q_0} + h(0)| \ll 1 \quad (57)$$

Hence tsunami-wave initial states that satisfy the tsunami slow-roll condition (57) *lead to an inflationary stage*.

Moreover, in order to have slow roll (56) at later times, the effective friction coefficient $3h(\tau)$ should be larger than the square of the frequency in the evolution equations (41). That is,

$$\frac{q_0^2 + 1 + g\Sigma(0)}{3h(0)} \ll 1. \quad (58)$$

(i.e. the q_0 -mode should be deep inside the overdamped oscillatory regime). Eq.(58) implies that $h(0) \gg 1$ and this together with eq.(57) implies that δ_{q_0} must be negative.

A remarkable aspect of the narrow tsunami state is that it leads to a dynamical evolution of the metric similar to that obtained in *classical chaotic inflationary scenarios* in the slow roll approximation[3, 5]. In particular the expression for the acceleration (54) and the tsunami slow roll condition (56) are indeed similar to those obtained in classical chaotic inflationary models driven by a homogeneous classical field (zero mode). However, despite the striking similarity with classical chaotic models, we haste to add that both the conditions that define a tsunami state and the tsunami slow roll condition (56) guaranteed by the initial value (57) is of purely quantum mechanical origin in contrast with the classical chaotic-slow-roll scenario. Furthermore, we recall that the expectation value of the scalar field vanishes in this state.

3.1.1 Early time dynamics

Under the assumption of a tsunami wave initial state and the tsunami slow-roll condition (56) the contribution $\dot{\phi}_{q_0}(\tau)$ in the energy and in the pressure [see eq. (53)] can be neglected provided,

$$|\dot{\phi}_{q_0}(\tau)|^2 \ll \frac{q_0^2}{3a^2(\tau)} |\phi_{q_0}(\tau)|^2. \quad (59)$$

We call τ_A the time scale at which this rely no longer holds. Furthermore, we can approximate $\phi_{q_0}(\tau)$ by $\phi_{q_0}(0)$ if

$$\tau_A \ll \left| \frac{\phi_{q_0}(0)}{\dot{\phi}_{q_0}(0)} \right|. \quad (60)$$

This condition is fulfilled at least for $\tau_A \lesssim 1$ due to the tsunami slow-roll condition (56).

During this interval the Friedmann equation (52) takes the form,

$$\left[\frac{\dot{a}(\tau)}{a(\tau)} \right]^2 = L^2 \left[\frac{1}{2} |\phi_{q_0}(0)|^2 + \frac{1}{4} |\phi_{q_0}(0)|^4 + \frac{q_0^2}{2a^2(\tau)} |\phi_{q_0}(0)|^2 \right] = \frac{D}{a^2(\tau)} + E. \quad (61)$$

where we used that $g\Sigma(0) = |\phi_{q_0}(0)|^2$. This equation is valid as long as the characteristic time scale of variation of the metric is shorter than that of the mode $\phi_{q_0}(\tau)$.

The preceding equation can be integrated with solution

$$\begin{aligned} a(\tau) &= \sqrt{\frac{D}{E}} \sinh(\sqrt{E}\tau + c) \quad , \quad \frac{\ddot{a}(\tau)}{a(\tau)} = E > 0 \quad , \\ h(\tau) &= \sqrt{E} \coth(\sqrt{E}\tau + c) \quad , \quad \dot{h}(\tau) = -\frac{E}{\sinh^2(\sqrt{E}\tau + c)} \quad , \end{aligned} \quad (62)$$

where the constants D , E and c are given by,

$$D = L^2 \frac{q_0^2}{2} g\Sigma_0 \quad , \quad E = L^2 \left(\frac{g\Sigma_0}{2} + \frac{g\Sigma_0^2}{4} \right) \quad , \quad \sinh c = \sqrt{\frac{E}{D}} \quad , \quad (63)$$

and $g\Sigma_0 \equiv g\Sigma(0) = |\phi_{q_0}(0)|^2$.

We see from eqs. (62) that during this interval there is an inflationary stage with an accelerated expansion $\frac{\ddot{a}(\tau)}{a(\tau)} = E > 0$. We also see that $h(\tau)$ decreases with time until it reaches the constant value \sqrt{E} that determines the onset of a quasi-De Sitter inflationary stage.

We now estimate the range of validity of the solution in eqs. (62). The first condition in eq. (59) is more stringent than the second one in eq. (60) for most of the interesting range of parameters.

τ_A determines the time scale at which the solution (62) ceases to be valid, and the condition (59) is no longer fulfilled, i.e.,

$$|\dot{\phi}_{q_0}(\tau_A)|^2 \sim \frac{q_0^2}{3a^2(\tau_A)} |\phi_{q_0}(\tau_A)|^2 \quad . \quad (64)$$

When this equation is valid, we see from eq. (53) that $p + \epsilon$ becomes of the order of $|\dot{\phi}_{q_0}(\tau_A)|^2$. Furthermore, the above condition together with the slow roll condition, eq. (56), leads to

$$\frac{q_0^2}{a^2(\tau_A)} \ll 1 \quad . \quad (65)$$

Therefore, from eq.(61) we see that $h(\tau_A) \sim \sqrt{E}$ and is slowly varying.

Since the slow roll condition guarantees that $|\dot{\phi}_q(\tau_A)| \ll |\phi_q(\tau_A)|$, we can now use eq.(49) along with the evolution equation (51) which setting $h = \sqrt{E}$ leads to the following relation

$$\dot{\phi}_{q_0}(\tau_A) \simeq -\frac{1 + g\Sigma_0}{3\sqrt{E}} \phi_{q_0}(\tau_A) \quad . \quad (66)$$

From eq.(64) the scale τ_A is therefore estimated to be given by

$$\tau_A \sim \frac{1}{\sqrt{E}} \left[\text{ArgSinh} \left(\frac{\sqrt{3} q_0 E}{\sqrt{D}(1 + g\Sigma_0)} \right) - c \right] = \frac{1}{\sqrt{E}} \left[\text{ArgSinh} \left(\frac{L\sqrt{3g\Sigma_0}(1 + g\Sigma_0/2)}{\sqrt{2}(1 + g\Sigma_0)} \right) - c \right] \quad . \quad (67)$$

This initial inflationary period with a decreasing Hubble parameter exists provided the r.h.s. is here positive, i.e.,

$$q_0 > \frac{1 + g\Sigma_0}{\sqrt{3E}} \quad . \quad (68)$$

In order to distinguish this phase from the later stages, to be described below, we refer to this early time inflationary stage as ‘tsunami-wave inflation’ because the distinct evolution of the scale factor during this stage is consequence of the tsunami-wave properties.

At $\tau = \tau_A$ we have:

$$\begin{aligned} \phi_{q_0}(\tau_A) &\simeq \phi_{q_0}(0) = \sqrt{g\Sigma_0} & , & & \dot{\phi}_{q_0}(\tau_A) &\simeq -\frac{1+g\Sigma_0}{3h(\tau_A)} \phi_{q_0}(\tau_A) , \\ a(\tau_A) &\simeq \frac{\sqrt{3E}q_0}{1+g\Sigma_0} & , & & h(\tau_A) &\simeq \sqrt{E} . \end{aligned} \quad (69)$$

For $\tau > \tau_A$, $q_0/a(\tau) \ll 1$ and the physical wavevectors in the excited band have red-shifted so much that all terms containing q_0 become negligible in the evolution equations. Therefore, all modes in the excited band evolve as an *effective* $q = 0$ mode. Hence for $\tau > \tau_A$ the dynamics of the scale factor is described by an effective homogeneous zero mode and describes a different regime from the one studied above. Such regime is akin to the classical chaotic scenario.

3.1.2 The effective classical chaotic inflationary epoch

For $\tau \gg \tau_A$, when $q_0^2/a^2 \ll |\dot{\phi}_q|^2/|\phi_q|^2 \ll 1$ all the physical momenta corresponding to the comoving wavevectors in the excited band have redshifted to become negligible in the equations of motion. The dynamics is now determined by the following set of equations for the effective zero mode and the scale factor,

$$\begin{aligned} \ddot{\phi}_{q_0}(\tau) + 3h(\tau)\dot{\phi}_{q_0}(\tau) + [1 + |\phi_{q_0}(\tau)|^2] \phi_{q_0}(\tau) &= 0 , \\ h^2(\tau) &= L^2 \epsilon(\tau) , \end{aligned} \quad (70)$$

where the energy and pressure are given by,

$$\begin{aligned} \epsilon(\tau) &= \frac{1}{2} |\dot{\phi}_{q_0}(\tau)|^2 + \frac{1}{2} |\phi_{q_0}(\tau)|^2 + \frac{1}{4} |\phi_{q_0}(\tau)|^4 , \\ (p + \epsilon)(\tau) &= |\dot{\phi}_{q_0}(\tau)|^2 . \end{aligned} \quad (71)$$

The initial conditions on ϕ_{q_0} and $\dot{\phi}_{q_0}$ are determined by their values at the time τ_A , while the slow-roll condition (56) determines that the *imaginary* parts of ϕ_{q_0} and $\dot{\phi}_{q_0}$ are negligible.

Therefore, after τ_A the dynamic is identical to that of a classical homogeneous field (zero mode)

$$\eta_{eff}(\tau) = \text{Re}[\phi_{q_0}(\tau)] , \quad (72)$$

that satisfies the equations of motion,

$$\begin{aligned} \ddot{\eta}_{eff} + 3h\dot{\eta}_{eff} + (1 + \eta_{eff}^2) \eta_{eff} &= 0 , \\ h^2(\tau) &= L^2 \epsilon(\tau) . \end{aligned} \quad (73)$$

with energy and pressure,

$$\epsilon(\tau) = \frac{1}{2} \dot{\eta}_{eff}^2 + \frac{1}{2} \eta_{eff}^2 + \frac{1}{4} \eta_{eff}^4 ,$$

$$(p + \epsilon)(\tau) = \dot{\eta}_{eff}^2, \quad (74)$$

and initial conditions [using eq. (69)],

$$\begin{aligned} \eta_{eff}(\tau_A) &= \phi_{q_0}(\tau_A) = \phi_{q_0}(0) = \sqrt{g\Sigma_0}, \\ \dot{\eta}_{eff}(\tau_A) &= \dot{\phi}_{q_0}(\tau_A) = -\frac{1 + g\Sigma_0}{3h(\tau_A)} \eta_{eff}(\tau_A). \end{aligned} \quad (75)$$

Where the value of $\dot{\eta}_{eff}(\tau_A)$ is determined by the slow roll condition ($\Rightarrow \ddot{\phi}_{q_0}(\tau_A) \simeq 0$), the evolution eq. (51) and eq. (65) and $a(\tau_A)$ and $h(\tau_A)$ are given by eq. (69).

When $g\Sigma_0 \ll 1$, the quadratic term in the potential dominates, and we can integrate the previous equations to obtain

$$\eta_{eff}(\tau) = \eta_{eff}(\tau_A) - \frac{\sqrt{2}}{3L}(\tau - \tau_A), \quad (\text{for } g\Sigma_0 \ll 1). \quad (76)$$

This evolution is similar to that of classical chaotic inflationary models[3, 5]. Therefore for $\tau > \tau_A$ when the physical momenta in the excited band have redshifted so much that their contribution in the equations of motion of the quantum modes and the energy and pressure become negligible, the evolution of the quantum modes and the metric is akin to a classical chaotic inflationary scenario driven by a homogeneous c-number scalar field. This equivalence allows us to use the results obtained for classical chaotic inflation. Thus, as the classical slow roll condition ($|\dot{\eta}_{eff}| \ll |\eta_{eff}|$) holds, the evolution of the effective scalar field is overdamped and the system enters a quasi-De Sitter inflationary epoch. This inflationary period ends when the slowly decreasing Hubble parameter becomes of the order of the inflaton mass, i.e, $3h \sim 1 + \eta_{eff}^2$. At this stage the effective classical field exits the overdamped regime and starts to oscillate, the slow roll condition no longer holds and a matter dominated epoch ($|\dot{\eta}_{eff}| \sim |\eta_{eff}| \Rightarrow p \sim 0$) follows.

However, we emphasize that while the effective zero mode η_{eff} obeys a classical equation of motion and that the components of the energy momentum tensor eq.(74) are those from a classical field, the origin of this mode is *purely quantum mechanical*. From the identification (75) and eq. (49) it is clear that the effective zero mode is a collective superposition of modes in the highly excited band. From the initial and tsunami wave conditions $f_{q_0}(\tau_A) \sim 1/\sqrt{\Omega_{q_0}} \sim 1/\sqrt{g}$ it follows that the amplitude of the effective zero mode is $\eta_{eff}(\tau_A) \sim q_0\sqrt{\Delta q}$. Restoring the dimensions and the proper powers of the coupling that were absorbed in the constant L in the Friedmann equation we find that the equations of motion (73) with the stress tensor components (74) are those obtained from a (dimensionfull) classical homogeneous field $\varphi_{eff}(t)$ with a classical potential $V(\varphi_{eff})$ given by

$$\begin{aligned} \varphi_{eff}(t) &= \frac{m\sqrt{2}}{\sqrt{\lambda}} \eta_{eff}(\tau) \\ V(\varphi_{eff}) &= \frac{m^2}{2} \varphi_{eff}^2 + \frac{\lambda}{8} \varphi_{eff}^4 \end{aligned} \quad (77)$$

with the initial value at the time $t_A = \tau_A/m$

$$\varphi_{eff}(t_A) \sim \frac{m}{\sqrt{\lambda}} \sqrt{\frac{k_0^2 \Delta k}{m^3}} \quad (78)$$

The non-perturbative amplitude of the effective zero mode is a consequence of the non-perturbative amplitude of the excited *quantum* modes with an $\mathcal{O}(1/\lambda)$ number of quanta.

3.1.3 Number of e-folds

An important cosmological quantity is the total number of e-folds during inflation. As discussed above, there are two different inflationary stages, the first one is determined by the equations (61)-(62) and characterized by a rapid fall-off of the Hubble parameter approaching a quasi-De Sitter stage. This new stage has been referred to as the tsunami-wave inflationary stage above to emphasize that the dynamics is determined by the distinct characteristics of the tsunami-wave initial stage.

The second stage is described by an effective zero mode and the evolution equations (73, 74) and is akin to the chaotic inflationary stage driven by a classical homogeneous scalar field. The crossover between the two regimes is determined by the time scale τ_A (in units of the inflaton mass) and given by eqs.(67) at which the contribution from the term $q_0^2/a^2(\tau)$ to the equations of motion becomes negligible. Therefore there are two distinct contributions to the total number of e-folds, which is given by

$$N_e(q_0, h(0)) = \log a(\tau_A) + N_e(0, h(\tau_A)) , \quad (79)$$

where $a(\tau_A)$ is given by eq. (75) and $N_e(0, h(\tau_A))$ is just the number of e-folds for classical chaotic inflation with an initial Hubble parameter $h(\tau_A)$.

We can express $h(\tau_A)$ as a function of q_0 and $h(0)$,

$$h(\tau_A) = L \sqrt{\frac{g\Sigma_0}{2} + \frac{(g\Sigma_0)^2}{4}} = \sqrt{h^2(0) - L^2 \frac{q_0^2}{2} g\Sigma_0} . \quad (80)$$

The number of e-folds during the first stage, is given by

$$\log a(\tau_A) \sim \log \left[\frac{\sqrt{3E} q_0}{1 + g\Sigma_0} \right] \quad (81)$$

The expression for the number of e-folds during the following, chaotic inflationary stage simplifies when $g\Sigma_0 \ll 1$. In this case the quadratic term in the potential dominates, and we can obtain simple analytical expressions

$$N_e(0, h(\tau_A)) = \frac{3L^2}{4} \eta_{eff}^2(\tau_A) = \frac{3L^2}{4} g\Sigma_0 = \frac{3L^2}{2} \frac{\epsilon_0}{1 + q_0^2} , \quad (\text{for } g\Sigma_0 \ll 1) . \quad (82)$$

We see that the number of efolds grow when q_0 decreases at fixed initial energy ϵ_0 . That is, we have more efolds when the energy is concentrated at low momenta.

3.1.4 In summary

Before proceeding to a full numerical study of the evolution we summarize the main features of the dynamics gleaned from the narrow tsunami case to compare with the numerical results.

- The conditions for tsunami-wave inflation are *i)* a band of excited states centered at a momentum k_0 with a non-perturbatively large $\mathcal{O}(1/g)$ number of quanta in this band, and

ii) the tsunami slow-roll condition eq.(56). These conditions are guaranteed by the initial conditions on the mode functions given by eq.(42) with the tsunami-wave distributions of the general form given by eqs. (46), (57).

- There are two successive inflationary periods. During the first one, described in sec. 3.1.1, the dynamics is completely characterized by the distinct features of the tsunami-wave initial state, the Hubble parameter falls off fast and reaches an approximately constant value \sqrt{E} that characterizes the quasi-De Sitter epoch of inflation of the second period. The second stage, described in sec. 3.1.2 can be described in terms of an effective classical zero mode and the evolution of this effective mode and that of the Hubble parameter are akin to the standard chaotic inflationary scenario.
- The tsunami-wave initial state can be interpreted as a *microscopic* justification of the classical chaotic scenario described by an effective classical zero mode of large amplitude. The amplitude of this effective zero mode is *non-perturbative* as a consequence of the non-perturbative $\mathcal{O}(1/\lambda)$ number of quanta in the narrow band of excited modes. Thus the initial value of the effective, classical zero mode that describes the second, chaotic inflationary stage, is completely determined by the quantum initial state.
- An important point from the perspective of structure formation is that the band of excited wavevectors centered at q_0 either correspond to superhorizon modes initially, or all of the excited modes cross the horizon during the first stage of inflation, i.e, during the tsunami stage. This is important because the chaotic second stage of inflation which dominates during a longer period guarantees that the band of excited modes have become superhorizon well before the last 10 e-folds of inflation and hence cannot affect the power spectrum of the temperature anisotropies in the CMB. The fact that the tsunami-wave initial state is such that the very high energy modes (necessarily trans-Planckian) that cross the horizon during the last 10 e-folds and are therefore of cosmological importance today are in their (conformal) vacuum state leads to the usual results from chaotic inflation for the power spectrum of scalar density perturbations.

Although these conclusions are based on the narrow tsunami case, we will see below that a full numerical integration of the self-consistent set of equations of motion confirms this picture.

In sections 3.3 and 4 we show how this results can be easily extended to *more general particle distributions* and *more general initial states*.

3.2 Numerical example

To make contact with familiar models of inflation with an inflaton field with a mass near the grand unification scale, we choose the following values of the parameters:

$$\frac{m}{M_{Pl}} = 10^{-4} \ , \ \lambda = 10^{-12} \ , \ N = 20 \tag{83}$$

where the number of scalar fields $N = 20$ has been chosen as a generic representative of a grand unified quantum field theory. For these values we find

$$L^2 \equiv \frac{16 \pi N m^2}{3 M_{Pl}^2 \lambda} = 3.35 \cdot 10^6 \ .$$

As an example we shall consider an initial energy density $\rho_0 = \langle T_{00} \rangle = 10^{-2} M_{Pl}^4$. Thus, the initial value for the Hubble parameter is $H_0 = \sqrt{8\pi\rho_0/3M_{Pl}} = 3.53 \cdot 10^{18} \text{ GeV} (= 1.654 \cdot 10^{52} \text{ km/s/Mpc})$. These initial conditions in dimensionless variables give $\epsilon_0 = 2.50$ and $h(0) = 2890$.

In addition, the slow roll conditions (58) imply:

$$\frac{q_0^2 + 1 + g\Sigma_0}{3h(0)} \ll 1$$

which in this case results in

$$q_0 \ll 95.$$

We choose $q_0 = 80.0$, and initial conditions in eq. (42) with Ω_q and δ_q given by eq. (46) and (57). These initial conditions satisfy the tsunami slow roll condition,

$$|\omega_q \delta_q + h(0)| \ll 1 \quad (84)$$

Furthermore, we take $\Delta q = 0.1$ and \mathcal{N}_Ω is adjusted by fixing the value $g\Sigma(0) = g\Sigma_0$ which for the values chosen for ϵ_0 and by eqs.(39) and (49) and (56) gives $g\Sigma_0 = 7.81 \cdot 10^{-4}$.

Figure 1 displays ϵ_0 vs q_0 along lines of constant number of e-folds, while figures 2-7 display the solution of the full set of equations (41)-(44) with (39). An important feature that emerges from these figures is that for the set of parameters that are typical for inflationary scenarios and for large values of $q_0 = k_0/m$ (but well below the Planck scale) the number of e-folds obtained is more than sufficient as shown by fig.6.

We also show that the dynamics of the full set of equations (41)-(44) with (39) is correctly approximated by the narrow tsunami case studied in the previous subsections: the one mode approximation [eqs. (49)-(53)], the early time analytical formulae (for $\tau \leq \tau_A$) [eqs. (62)-(63)], and the effective classical field (for $\tau > \tau_A$) [eqs. (74)]. The agreement between the analytic treatment and the full numerical evolution is displayed in figures 2-7.

The early time analytic expressions predict an inflationary period during which the Hubble parameter falls off fairly fast, that lasts up to $\tau_A \sim 0.133$ [eq. (67)] reaching an asymptotic value of $h(\tau_A) = 36.2$ [eqs. (63) and (69)]. The one mode approximation gives the same prediction $h(\tau_A) = 36.1$, and numerically evolving the full set of equations we find $h(\tau_A) = 35.5$. Thus, we see from this values and from figs. 2 and 3 that both approximations are fairly accurate for early times.

After τ_A , the geometry reaches a quasi-De Sitter epoch. We have shown in the previous subsection that after the time τ_A the evolution equations for the one mode approximation reduce to those of an effective classical field. The effective zero mode approximation correctly predicts the dynamics in this epoch as can be gleaned from figures 3-6.

While the stage of early tsunami inflation up to τ_A results in only 8.5 efolds, the following quasi-de Sitter stage described by the effective classical scalar field lasts for a total of 1900 efolds. For the values of parameters chosen above, $g\Sigma_0 \ll 1$, hence we can estimate the number of efolds with eq. (82). Using eq. (79) we obtain a total of 1970 efolds while the one mode approximation yields 1960 efolds. Both results agree with the full numerical solution of the equations (see fig. 6).

Furthermore, as stated above inflation ends when $h \sim \frac{1+\eta_{eff}^2}{3} \sim \frac{1}{3}$, after which a matter dominated epoch follows.

3.3 Other distributions and other states:

The validity of the physical picture that emerges from the previous analytic and numerical study is not restricted to pure states or narrow distributions of the form given by (46). We have also studied more general distributions and mixed states:

Other distributions:

The narrow tsunami case where a single quantum mode q_0 dominates the dynamics has been extremely useful to study the dynamics in the previous section. The generalization to the case with continuous distributions of q -modes can be easily obtained making the changes:

$$\begin{aligned} |\phi_{q_0}(\tau)|^2 &\rightarrow g \int q^2 dq |f_q(\tau)|^2 , \\ |\dot{\phi}_{q_0}(\tau)|^2 &\rightarrow g \int q^2 dq |\dot{f}_q(\tau)|^2 , \\ q_0^2 |\phi_{q_0}(\tau)|^2 &\rightarrow g \int q^2 dq q^2 |f_q(\tau)|^2 . \end{aligned} \quad (85)$$

The two stages of inflation are always present for such continuous modes distribution as long as the following generalized slow-roll conditions is fulfilled

$$g \int q^2 dq |\dot{f}_q(\tau)|^2 \ll g \int q^2 dq |f_q(\tau)|^2 \quad (86)$$

that imposes on δ_q the condition $|\omega_q \delta_q + h(0)| \ll 1$.

The effective zero mode in the second stage of inflation is now given by

$$\eta_{eff}^2(\tau) = g \int q^2 dq |f_q(\tau)|^2 .$$

Our numerical study with general distributions reveals that the analytical picture obtained by substituting eq.(85) in sec. IIIA correctly reproduce the dynamics.

Other (mixed) states: Although we have focused for simplicity on tsunami pure initial states, we have also investigated the possibility of mixed states. Mixed state density matrices and their time evolution are discussed in the appendix. The mixing can be parametrized in terms of angles Θ_k as given in equation (105) and the number of (conformal) quanta are given by eq. (119). The *only* relevant changes that occur are in the integrals for $\Sigma(\tau)$; $\varepsilon(\tau)$; $p(\tau)$ in which

$$|f_q(\tau)|^2 \rightarrow |f_q(\tau)|^2 \coth \left[\frac{\Theta_q}{2} \right] , \quad |\dot{f}_q(\tau)|^2 \rightarrow |\dot{f}_q(\tau)|^2 \coth \left[\frac{\Theta_q}{2} \right] \quad (87)$$

Tsunami and slow-roll conditions on the mode functions given by eqs. (30) and (57) lead to tsunami-wave inflation followed by chaotic inflation just as discussed above. In the narrow tsunami case the only change is that the effective q_0 -mode is rescaled by the mixing factor, i.e,

$$|\phi_{q_0}(\tau)|^2 \rightarrow |\phi_{q_0}(\tau)|^2 \coth \left[\frac{\Theta_{q_0}}{2} \right] , \quad |\dot{\phi}_{q_0}(\tau)|^2 \rightarrow |\dot{\phi}_{q_0}(\tau)|^2 \coth \left[\frac{\Theta_{q_0}}{2} \right] \quad (88)$$

It is also illuminating to contrast the tsunami-wave mixed states with the more familiar *thermal* mixed states. The latter are obtained by the choice

$$\Omega_q = \omega_q \quad ; \quad \delta_q = 0 \quad ; \quad \Theta_q = \frac{\omega_q}{T} \quad (89)$$

with T some value of temperature. In this case it is straightforward to see that (quantum) equipartition results in that the contributions of the modes and their time derivatives to the energy and pressure are of the same order ($|\dot{f}_q(\tau)|^2 \sim [h(0)^2 + \omega_q^2] |f_q(\tau)|^2$). Hence, for these thermal mixed states the tsunami slow-roll condition is not fulfilled. This is obviously *not* surprising, such a choice of thermal mixed state leads to a FRW epoch which is not inflationary. Hence the tsunami-wave initial conditions along with the generalized slow-roll conditions lead to *two* successive inflationary epochs in striking contrast to the familiar mixed thermal states.

4 Generalized chaotic inflation

The previous analysis, confirmed by the numerical evolution of the full self-consistent set of equations leads to one of the important conclusions of this article, that tsunami-wave initial states provide a microscopic justification of the chaotic inflationary scenario.

We have focused our discussion on initial states with vanishing expectation value of the scalar field (order parameter) and where the energy is concentrated in a momentum band (tsunami initial states). This choice brings to the fore the striking contrast between this novel *quantum state* and the usual classical approach to chaotic inflation. In this section we study the dynamics in the case in which the initial state allows for a non-vanishing expectation value of the scalar field *along* with some of the initial energy localized in excited quanta. We refer to this case as generalized chaotic inflation to distinguish from the tsunami-wave state studied above. This generalization thus includes both cases: the classical chaotic inflation in the limit when there are no excited modes, as well as the tsunami initial state when all of the energy is localized in a band of excited modes and the expectation value of the field vanishes.

The relevant equations of motion in comoving time for the mode functions in this case are given by (121) in the appendix. Along with the dimensionless variables (37) it is also convenient to introduce a dimensionless expectation value as

$$\eta^2(\tau) = \frac{\lambda}{2m^2} \phi^2(t) \quad (90)$$

In this generalized case with $\eta \neq 0$ the equations of motion for the mode functions $f_q(\tau)$ (in terms of dimensionless variables) are the same as in eq. (41) after the replacement $g\Sigma(\tau) \rightarrow g\Sigma(\tau) + \eta^2(\tau)$ and the equation of motion for $\eta(\tau)$ is given by

$$\frac{d^2\eta(\tau)}{d\tau^2} + 3h(\tau)\frac{d\eta(\tau)}{d\tau} + [1 + \eta^2(\tau) + g\Sigma(\tau)] \eta(\tau) = 0 \quad (91)$$

$$\eta(0) = \eta_0 \quad ; \quad \dot{\eta}(0) = \dot{\eta}_0 \quad (92)$$

The Einstein-Friedmann equation is given by (44) but with the energy density and pressure now given by

$$\begin{aligned} \epsilon(\tau) &= \frac{1}{2}\dot{\eta}^2 + \frac{1}{2}(g\Sigma + \eta^2) + \frac{1}{4}(g\Sigma + \eta^2)^2 + \\ &+ \frac{g}{2} \int q^2 dq \left\{ |\dot{f}_q|^2 - S_1(q, \tau) + \frac{q^2}{a^2} [|f_q|^2 - S_2(q, \tau)] \right\} , \end{aligned} \quad (93)$$

$$(p + \epsilon)(\tau) = \dot{\eta}^2 + g \int q^2 dq \left\{ |\dot{f}_q|^2 - S_1(q, \tau) + \frac{q^2}{3a^2} [|f_q|^2 - S_2(q, \tau)] \right\} . \quad (94)$$

where the renormalization subtractions S_1 ; S_2 are obtained from those given by eq. (40) upon the replacement $g\Sigma \rightarrow g\Sigma + \eta^2$.

From this expression we see that for fixed (large) energy density Status: RO

there are two different possibilities: if the zero mode squared $\eta^2(0)$ is larger than the quantum fluctuations $g\Sigma$ and $g \int q^4 dq |f_q|^2$, the dynamics is basically similar to that in the usual chaotic inflationary scenarios. This corresponds to most of the initial energy density to be in the zero mode and little energy density in the band of excited states. On the other hand, for small $\eta^2(0)$ most of the initial energy density is in the tsunami quantum state and the initial dynamics is akin to the $\eta = 0$ case. To quantify this statement and clarify the interplay and crossover of behaviors between the $\eta = 0$ and the generalized chaotic case, we now resort again to the narrow tsunami case, which highlights the essential physics.

The relevant equations are: i) the equations of motion for the effective q_0 -mode (50),

$$\ddot{\phi}_{q_0}(\tau) + 3h(\tau)\dot{\phi}_{q_0}(\tau) + \left[\frac{q_0^2}{a^2(\tau)} + 1 + \eta^2(\tau) + |\phi_{q_0}(\tau)|^2 \right] \phi_{q_0}(\tau) = 0. \quad (95)$$

and for the zero mode η

$$\ddot{\eta}(\tau) + 3h(\tau)\dot{\eta}(\tau) + [1 + \eta^2(\tau) + |\phi_{q_0}(\tau)|^2] \eta(\tau) = 0 \quad (96)$$

and the Hubble parameter given by (52) with the energy density given by

$$\epsilon(\tau) = \frac{1}{2} \left(|\dot{\phi}_{q_0}(\tau)|^2 + \dot{\eta}^2(\tau) \right) + \frac{1}{2} \left(|\phi_{q_0}(\tau)|^2 + \eta^2(\tau) \right) + \frac{1}{4} \left(|\phi_{q_0}(\tau)|^2 + \eta^2(\tau) \right)^2 + \frac{q_0^2}{2a^2(\tau)} |\phi_{q_0}(\tau)|^2. \quad (97)$$

The acceleration of the scale factor is now given by

$$\frac{\ddot{a}(\tau)}{a(\tau)} = -L^2 \left[|\dot{\phi}_{q_0}(\tau)|^2 + \dot{\eta}^2(\tau) - \frac{1}{2} \left(|\phi_{q_0}(\tau)|^2 + \eta^2(\tau) \right) - \frac{1}{4} \left(|\phi_{q_0}(\tau)|^2 + \eta^2(\tau) \right)^2 \right] \quad (98)$$

where again we have neglected the renormalization contributions and terms of $\mathcal{O}(g)$ consistently in the weak coupling limit $g \ll 1$. From eq. (98) the generalized condition for an inflationary epoch (within the narrow tsunami case) becomes

$$|\dot{\phi}_{q_0}(\tau)|^2 + \dot{\eta}^2(\tau) < \frac{1}{2} \left(|\phi_{q_0}(\tau)|^2 + \eta^2(\tau) \right) + \frac{1}{4} \left(|\phi_{q_0}(\tau)|^2 + \eta^2(\tau) \right)^2 \quad (99)$$

which is fulfilled if the following *generalized slow roll condition* holds

$$|\dot{\phi}_{q_0}(\tau)|^2 + \dot{\eta}^2(\tau) \ll |\phi_{q_0}(\tau)|^2 + \eta^2(\tau) \quad (100)$$

Under these conditions and from eqs.(95)-(96) and the dynamics of the scale factor driven by the energy density eq.(97) we can now distinguish the following different inflationary scenarios.

- *Tsunami dominated:* When $(q_0^2 + 1) |\phi_{q_0}(0)|^2 \gtrsim \eta^2(0)$ the excited states in the tsunami-wave carry most of the initial energy density. In this case the results of the previous section apply and the scale factor takes the form as in eq.(61) with D given by eq.(63), and E given by

eq.(63) but with $g\Sigma_0 \rightarrow g\Sigma_0 + \eta^2(0)$. There are *two* consecutive inflationary stages as in the previous section. The first described by eq.(62), lasts up to the time scale τ_A defined by

$$\dot{\eta}^2(\tau_A) + |\dot{\phi}_{q_0}(\tau_A)|^2 \sim \frac{q_0^2}{3a^2(\tau_A)} |\phi_{q_0}(\tau_A)|^2$$

at which the red-shift of the momentum q_0 is such that $q_0/a(\tau_A) \ll 1$. The secondary stage is a usual classical chaotic inflationary epoch determined by the dynamics of an effective zero mode given by

$$\eta_{eff}^2(\tau) = \eta^2(\tau) + |\phi_{q_0}(\tau)|^2 \quad (101)$$

because for $\tau > \tau_A$, $q_0^2/a^2(\tau) \ll 1$ and the effective equation of motion for $\phi_{q_0}(\tau)$ is the same as that for $\eta(\tau)$.

- *Zero mode dominated:* When $\eta^2(0) \gg (q_0^2 + 1) |\phi_{q_0}(0)|^2$ the energy density stored in the zero mode is much larger than that contributed by the excited states in the tsunami-wave. In this case the energy density eq.(97) is completely dominated by the zero mode. The ensuing dynamics is the familiar classical chaotic scenario driven by a classical zero mode, *without* an early stage in which the scale factor is given by eq.(62) which is the hallmark of the tsunami-wave dynamics.

This analysis in the narrow tsunami case does highlight the important aspects of the dynamics in a clear manner, allowing a clean separation of the two cases described above. We have carried a full numerical integration of the equations of motion that reproduce the results described above. The criterion for the crossover between tsunami-wave and classical chaotic inflation is determined by the relative contributions to the energy density from the quantum fluctuations in the tsunami wave state as compared to the energy density of the zero mode .

The previous results [eqs.(90)-(101)] can be easily generalized for generic continuous distributions of modes and for mixed states. One has just to make the changes indicated in eq.(85) for generic distributions and in eq.(87) for mixed states.

The generalized slow-roll conditions takes then the form:

$$\dot{\eta}^2(\tau) + g \int q^2 dq |\dot{f}_q(\tau)|^2 \coth \left[\frac{\Theta_q}{2} \right] \ll \eta^2(\tau) + g \int q^2 dq |f_q(\tau)|^2 \coth \left[\frac{\Theta_q}{2} \right]$$

during the first stage of inflation.

The dynamics is tsunami dominated provided,

$$g \int q^2 dq (1 + q^2) |f_q(0)|^2 \coth \left[\frac{\Theta_q}{2} \right] \gtrsim \eta^2(0)$$

The effective zero mode in the second stage of inflation is now given by

$$\eta_{eff}^2(\tau) = \eta^2(\tau) + g \int q^2 dq |f_q(\tau)|^2 \coth \left[\frac{\Theta_q}{2} \right] .$$

These results have been verified by numerical integration of the full set of evolution equations (39)-(44).

5 Conclusions

In this article we have studied inflation in typical scalar field theories as a consequence of the time evolution of a novel quantum state. This quantum state is characterized by a *vanishing* expectation value of the scalar field, i.e, a vanishing zero mode, but a non-perturbatively large number of quanta in a momentum band, thus its name—tsunami-wave state.

This state leads to a non-perturbatively large energy density which is localized in the band of excited quantum modes. We find that the self-consistent equations for the evolution of this quantum state and the scale factor lead to inflation under conditions that are the quantum analog of slow-roll.

The self-consistent evolution was studied analytically and numerically in a wide range of parameters for the shape and position of the distribution of excited quanta. The numerical results confirm all the features obtained from the analytic treatment.

Under the conditions that guarantee inflation, there are two consecutive but distinct inflationary epochs. The first stage features a rapid fall-off of the Hubble parameter and is characterized by the quantum aspects of the state. During this first stage the large number of quanta in the excited band are redshifted and build up an *effective homogeneous classical condensate*. The amplitude of this condensate is non-perturbatively large, of $\mathcal{O}(1/\lambda)$, as a consequence of the non-perturbatively large number of quanta in the band of excited modes.

The second stage is similar to the classical chaotic scenario and it is driven by the dynamics of this effective classical condensate, with vanishing expectation value of the scalar field. Under the tsunami slow-roll conditions on the quantum state, the total number of e-folds is more than enough to satisfy the constraints of inflationary cosmology. The band of excited wave-vectors if not initially outside the causal horizon, becomes superhorizon during the first inflationary stage, therefore these excited states do not modify the power spectrum of scalar density perturbations on wavelengths that are of cosmological relevance today.

Therefore, these tsunami-wave quantum states provide a quantum field theoretical justification of chaotic (or in general large field) inflationary models and yield to a microscopic understanding of the emergence of classical homogeneous field configurations of large amplitude as an effective collective mode built from the large number of quanta in the excited band.

In addition, we recall that it is necessary to choose an initial state that breaks the $\Phi \rightarrow -\Phi$ symmetry in classical chaotic scenarios [3, 4]. This is *not* the case here. We have inflation with *zero* expectation value of the scalar field.

For completeness we have also studied more general states and established the important difference between tsunami (pure or mixed) quantum states leading to inflation, and thermal mixed states which do not lead to inflation.

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A Equations of motion in the large N limit and initial states.

In this appendix we obtain the equations of motion in conformal time for the generalized case in which the initial state is determined by a density matrix. The evolution of the functional density

matrix is given by the Liouville equation in conformal time

$$i \frac{\partial \rho}{\partial T} = [H, \rho] \implies i \frac{\partial}{\partial T} \rho[\vec{\Psi}, \vec{\Psi}; T] = \left(H \left[\frac{\partial}{\partial \vec{\Psi}}; \vec{\Psi} \right] - H \left[\frac{\partial}{\partial \vec{\Psi}}; \vec{\Psi} \right] \right) \rho[\vec{\Psi}, \vec{\Psi}; T] \quad (102)$$

where the Hamiltonian H is given by eq.(15) to leading order in the large N limit. Consistently with the fact that in the large N limit the Hamiltonian describes a collection of harmonic oscillators, we propose a Gaussian density matrix

$$\rho[\vec{\pi}, \vec{\pi}, T] = \mathcal{N}(T) \prod_k \exp \left\{ -\frac{A_k(T)}{2} \vec{\pi}_k \cdot \vec{\pi}_{-k} - \frac{A_k^*(T)}{2} \vec{\pi}_k \cdot \vec{\pi}_{-k} - B_k(T) \vec{\pi}_k \cdot \vec{\pi}_{-k} \right\} \quad (103)$$

The hermiticity condition $\rho^\dagger = \rho$ for the density matrix impose that B_k must be real. In addition, since $\vec{\pi}(\mathbf{x}, T)$ is a real field, its Fourier components must obey the hermiticity condition $\vec{\pi}_{-k}(T) = \vec{\pi}_k^*(T)$; thus, we can assume $A_{-k}(T) = A_k(T)$ without loss of generality.

The evolution equations for $A_k(T)$, $\mathcal{N}(T)$ and $B_k(T)$ are obtained from the Liouville eq. (102) where the hamiltonian is given by eq. (15). We find

$$iA_k' = A_k^2 - B_k^2 - a^2(T) \omega_k^2(T) \quad , \quad iB_k' = B_k (A_k - A_k^*) \\ \mathcal{N}_\rho(T) = \mathcal{N}_\rho(0) e^{-\frac{iN}{2} \int_0^T d\tilde{T} \sum_k [A_k(\tilde{T}) - A_k^*(\tilde{T})]} \quad , \quad (104)$$

where the prime denotes derivative with respect to conformal time T .

The normalization factor for mixed states $\mathcal{N}_\rho(T)$ is related with the normalization factor of pure states $\mathcal{N}_\Upsilon(T)$ by

$$\mathcal{N}_\rho(T) = \mathcal{N}_\Upsilon(T) \mathcal{N}_\Upsilon(T)^*$$

where

$$\mathcal{N}_\Upsilon(T) = \mathcal{N}_\Upsilon(0) \exp \left\{ -i \int_0^T dT' \left[NV h_{cl}(T') - \frac{\lambda}{8N} \left(\sum_k \langle \vec{\pi}_k \cdot \vec{\pi}_{-k} \rangle(T') \right)^2 + \frac{N}{2} \sum_k A_k(T') \right] \right\}$$

Writing A_k in terms of its real and imaginary parts $A_k = A_{R,k} + iA_{I,k}$, we find that $B_k/A_{R,k}$ is a conserved quantity. Thus, we can introduce without loss of generality the variables $\mathcal{A}_{R,k}(T)$, $\mathcal{A}_{I,k}(T)$ and Θ_k defined by

$$A_{R,k}(T) \equiv \mathcal{A}_{R,k}(T) \coth \Theta_k \quad , \quad A_{I,k}(T) \equiv \mathcal{A}_{I,k}(T) \\ B_k(T) \equiv -\frac{\mathcal{A}_{R,k}(T)}{\sinh \Theta_k} \quad (105)$$

where Θ_k is a time independent real function.

Introducing the complex variable

$$\mathcal{A}_k = \mathcal{A}_{R,k} + i\mathcal{A}_{I,k} \quad (106)$$

we see that it obeys the following Ricatti equation

$$i\mathcal{A}'_k = \mathcal{A}_k^2 - a^2(\mathcal{T}) \omega_k^2(\mathcal{T}) \quad (107)$$

This equation can be linearized defining

$$\mathcal{A}_k(\mathcal{T}) \equiv -i \frac{\varphi_k'^*(\mathcal{T})}{\varphi_k^*(\mathcal{T})} . \quad (108)$$

Then eq. (107) implies that the mode functions φ_k obey

$$\begin{aligned} \varphi_k'' + \omega_k^2(\mathcal{T}) \varphi_k &= 0 , \\ \omega_k^2(\mathcal{T}) &= k^2 + a^2(\mathcal{T}) \left[\mathcal{M}^2(\mathcal{T}) - \frac{\mathcal{R}(\mathcal{T})}{6} \right] , \end{aligned} \quad (109)$$

where $\mathcal{R}(\mathcal{T})$ is the Ricci scalar.

The relation (108) defines the mode functions $\varphi_k(\mathcal{T})$ up to an arbitrary multiplicative constant that we choose such that the wronskian takes the value,

$$\varphi_k \varphi_k'^* - \varphi_k' \varphi_k^* = 2i . \quad (110)$$

For this choice of the Wronskian the definition (108) becomes

$$\mathcal{A}_k = \frac{1}{|\varphi_k|^2} - \frac{i}{2} \frac{d}{d\mathcal{T}} \ln |\varphi_k|^2 . \quad (111)$$

The mass term in eq.(109) given by eq. (17) requires the self-consistent expectation value

$$\begin{aligned} \frac{\langle \vec{\pi}^2 \rangle_\rho}{N} &= \int \frac{d^3 k}{(2\pi)^3} \langle \vec{\pi}_k \cdot \vec{\pi}_{-k} \rangle_\rho \\ \langle \vec{\pi}_k \cdot \vec{\pi}_{-k} \rangle_\rho &= \text{Tr}_\rho \vec{\pi}_k \cdot \vec{\pi}_{-k} = \frac{1}{2[A_{R,k} + B_k]} = \frac{1}{2\mathcal{A}_{R,k}} \coth\left(\frac{\Theta_k}{2}\right) = \frac{1}{2} |\varphi_k|^2 \coth\left(\frac{\Theta_k}{2}\right) \end{aligned} \quad (112)$$

Thus, the evolution equations in terms of the mode functions are given by eq. (109) with

$$\mathcal{M}^2(\mathcal{T}) = m^2 + \xi \mathcal{R} + \frac{\lambda}{2} \frac{\psi^2}{a^2} + \frac{\lambda}{4} \int \frac{d^3 k}{(2\pi)^3} \frac{|\varphi_k|^2}{a^2} \coth\left(\frac{\Theta_k}{2}\right) \quad (113)$$

The evolution equation of the mode functions φ_k is the same as the Heisenberg equations of motion for the fields, hence we can write the Heisenberg field operators as

$$\vec{\pi}(\mathbf{x}, \mathcal{T}) = \int \frac{d^3 k}{\sqrt{2}(2\pi)^3} \left[\vec{a}_k \varphi_k(\mathcal{T}) e^{i\mathbf{k} \cdot \mathbf{x}} + \vec{a}_k^\dagger \varphi_k^*(\mathcal{T}) e^{-i\mathbf{k} \cdot \mathbf{x}} \right] \quad (114)$$

Thus, the definition (108) gives the relation between Schrödinger and Heisenberg pictures, since the functional density matrix (103) is in the Schrödinger picture.

The expectation value $\psi(\mathcal{T})$ [see eq. (13)] in conformal time obeys the following equation of motion[8]

$$\psi''(\mathcal{T}) + a^2(\mathcal{T}) \left[\mathcal{M}^2(\mathcal{T}) - \frac{\mathcal{R}(\mathcal{T})}{6} \right] \psi(\mathcal{T}) = 0 \quad (115)$$

$$\psi(0) = \psi_0 \quad ; \quad \psi'(0) = \psi'_0 \quad (116)$$

Hence, the evolution equations are given by (109), (113) and (115) with (116).

The initial density matrix in the Schrödinger picture is determined by specifying the initial values of $\mathcal{A}_{R,k}$, $\mathcal{A}_{I,k}$ and Θ_k . We will take $a(0) = 1$ and parameterize the initial value of \mathcal{A}_k as follows,

$$\mathcal{A}_{R,k}(0) = \Omega_k \quad , \quad \mathcal{A}_{I,k}(0) = \omega_k(0) \delta_k \quad (117)$$

The corresponding initial conditions for the mode functions are obtained from eq.(117) using eq.(108) and the Wronskian constraint eq.(110). These are given by

$$\varphi_k(0) = \frac{1}{\sqrt{\Omega_k}} \quad ; \quad \varphi'_k(0) = -[\omega_k(0) \delta_k + i\Omega_k] \varphi_k(0) \quad (118)$$

Defining the number of particles in terms of the adiabatic eigenstates of the Hamiltonian (15) as in eq. (27), it is straightforward to find that the initial occupation numbers are given by

$$n_k(0) = \langle \hat{n}_k(0) \rangle_{\rho(0)} = \frac{\Omega_k^2 + \omega_k^2(0) + \omega_k^2(0) \delta_k^2}{4\omega_k \Omega_k} \coth\left(\frac{\Theta_k}{2}\right) - \frac{1}{2} \quad (119)$$

For any mixing parameter $\Theta_k \neq 0$ the density matrix represents a *mixed state* since $B_k \neq 0$, a *pure* initial state is obtained by taking $\Theta_k = \infty$, in which case $B_k \rightarrow 0$ and the density matrix becomes a product of a wave functional times its complex conjugate.

It is convenient to pass to comoving time, this is achieved by the rescaling of the fields

$$\psi(\mathcal{T}(t)) = \phi(t) a(t) \quad , \quad \varphi_k(\mathcal{T}(t)) = f_k(t) a(t) \quad (120)$$

in terms of which the equations of motion are

$$\begin{aligned} \ddot{\phi}(t) + 3H(t) \dot{\phi}(t) + \mathcal{M}^2(t) \phi(t) &= 0 \\ \ddot{f}_k(t) + 3H(t) \dot{f}_k(t) + \left[\frac{k^2}{a^2(t)} + \mathcal{M}^2(t) \right] f_k(t) &= 0 \\ \mathcal{M}^2(t) = m^2 + \xi \mathcal{R}(t) + \frac{\lambda}{2} \phi^2(t) + \frac{\lambda}{4} \int \frac{d^3k}{(2\pi)^3} |f_k(t)|^2 \coth\left(\frac{\Theta_k}{2}\right) \end{aligned} \quad (121)$$

where the dots denote derivative with respect to the comoving time t . The initial conditions for the order parameter are its initial value $\phi(0)$, and its initial derivative $\dot{\phi}(0)$. For $a(0) = 1$, the initial conditions for the fluctuations are given by Θ_k and

$$f_k(0) = \frac{1}{\sqrt{\Omega_k}} \quad ; \quad \dot{f}_k(0) = -[\omega_k(0) \delta_k + H(0) + i\Omega_k] f_k(0) \quad (122)$$

[Those are the transformed of the initial conditions in conformal time eq. (118).]

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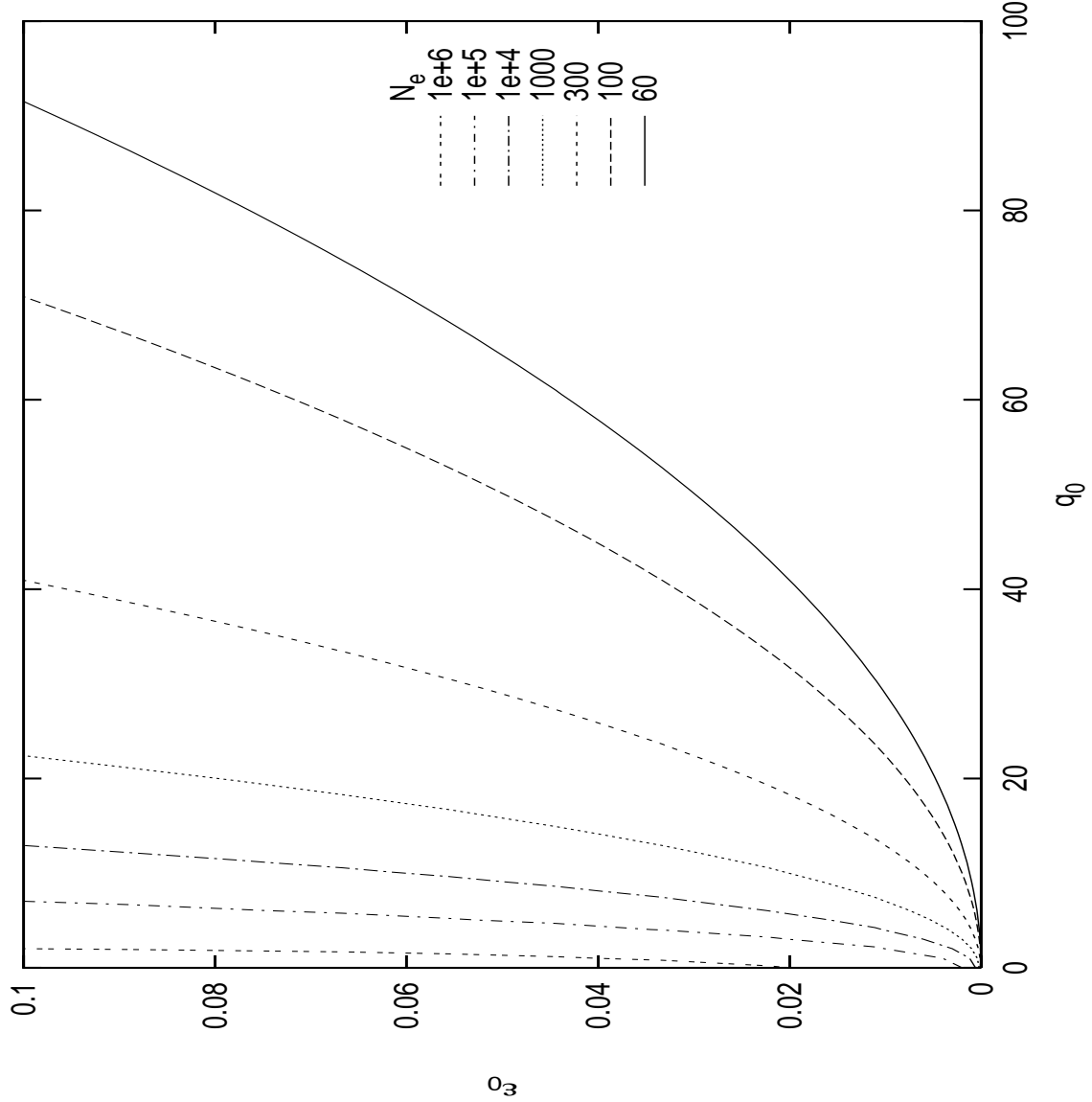
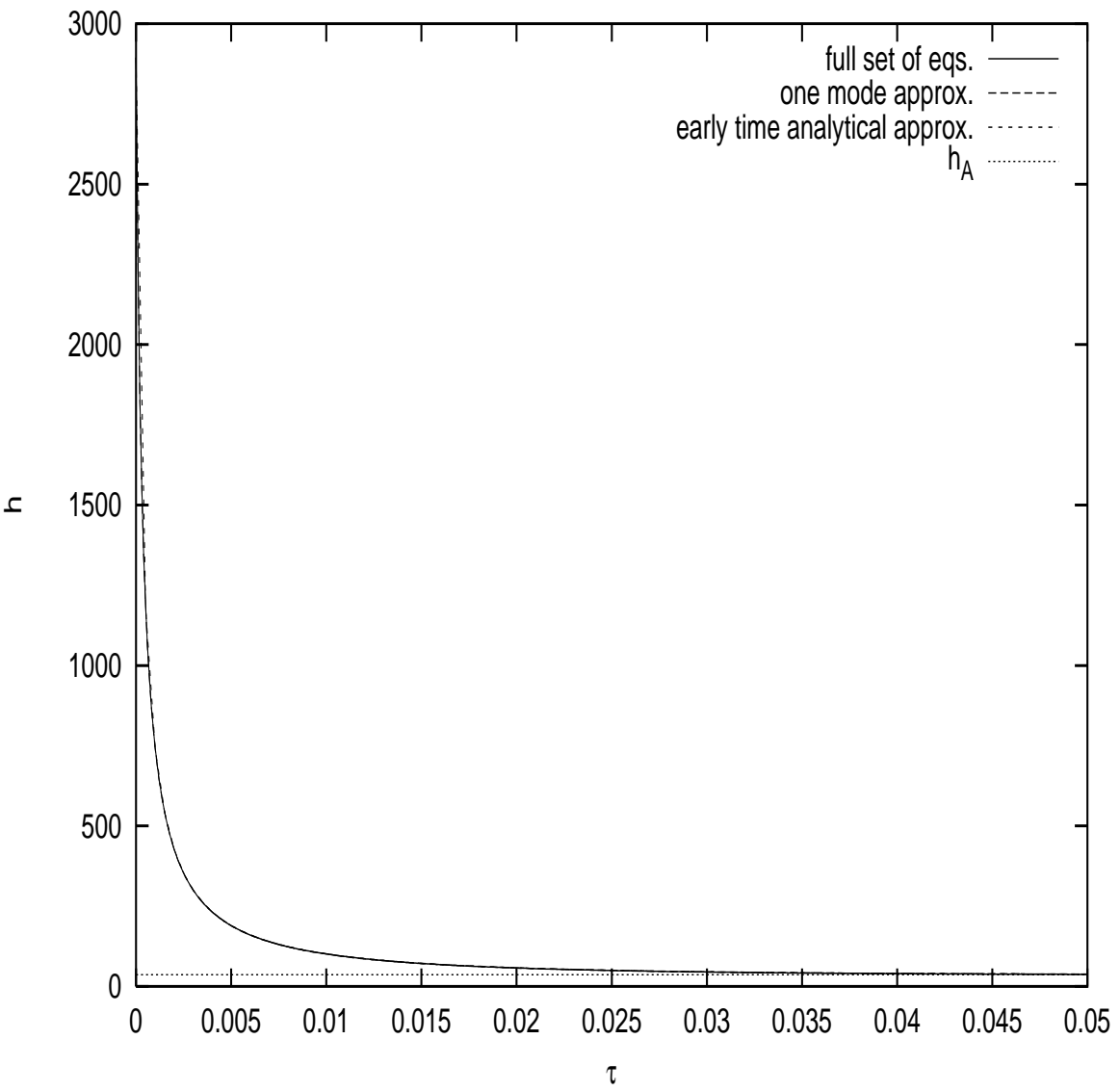


Figure 1: Tsunami inflation: isolines of constant number of efolds obtained from eq.(82) (valid for $g\Sigma_0 \ll 1$), for $m = 10^{-4}M_{Pl}$, $\lambda = 10^{-12}$ and $N = 20$.



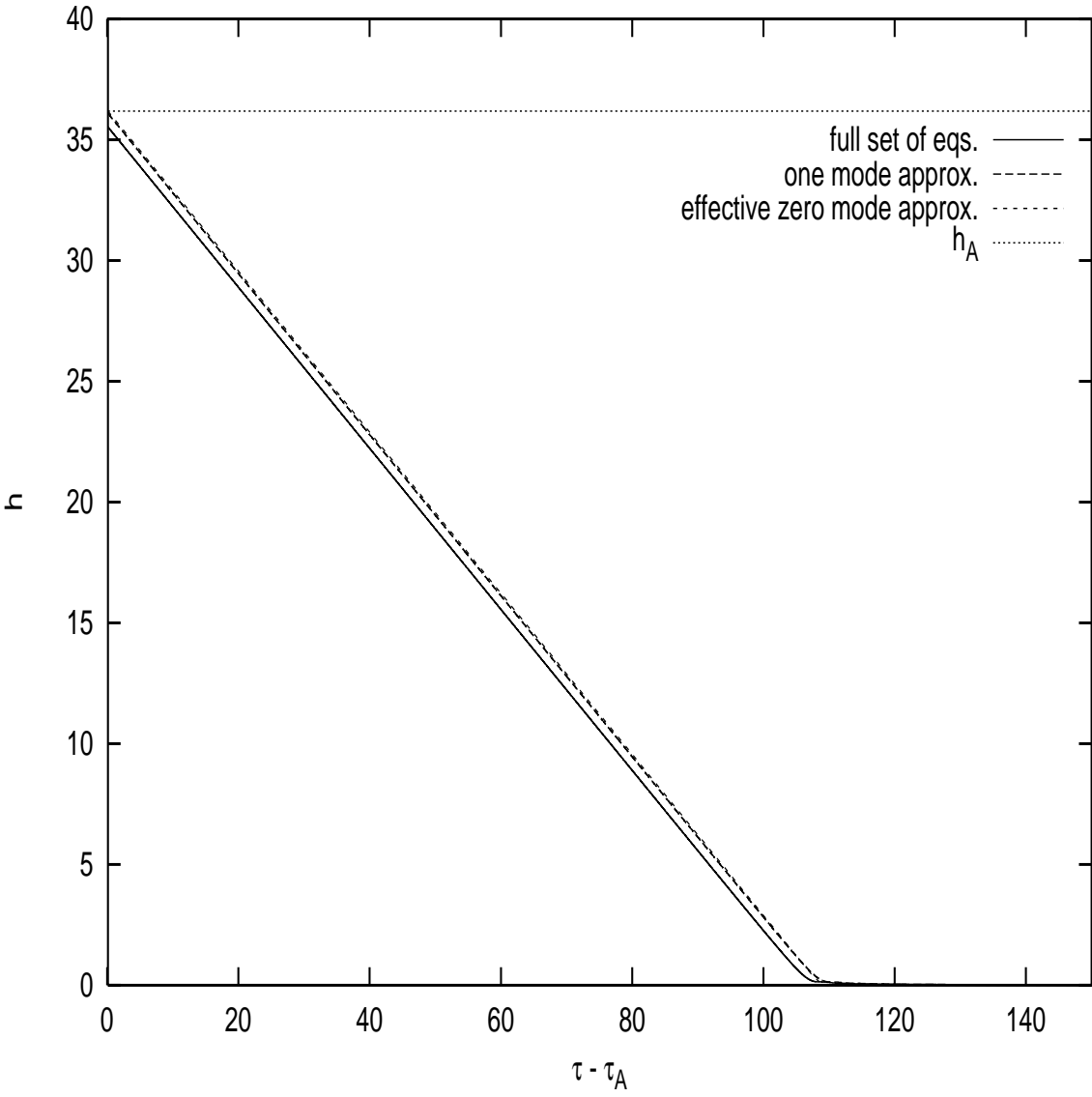


Figure 3: Tsunami inflation: $h(\tau)$ for $\tau > \tau_A$. The early time analytic approximation gives $h_A = 36.1$ (also with the one mode approx.), numerically we obtain $h(\tau_A) = 35.5$. Same parameters and initial conditions as in fig. 2.

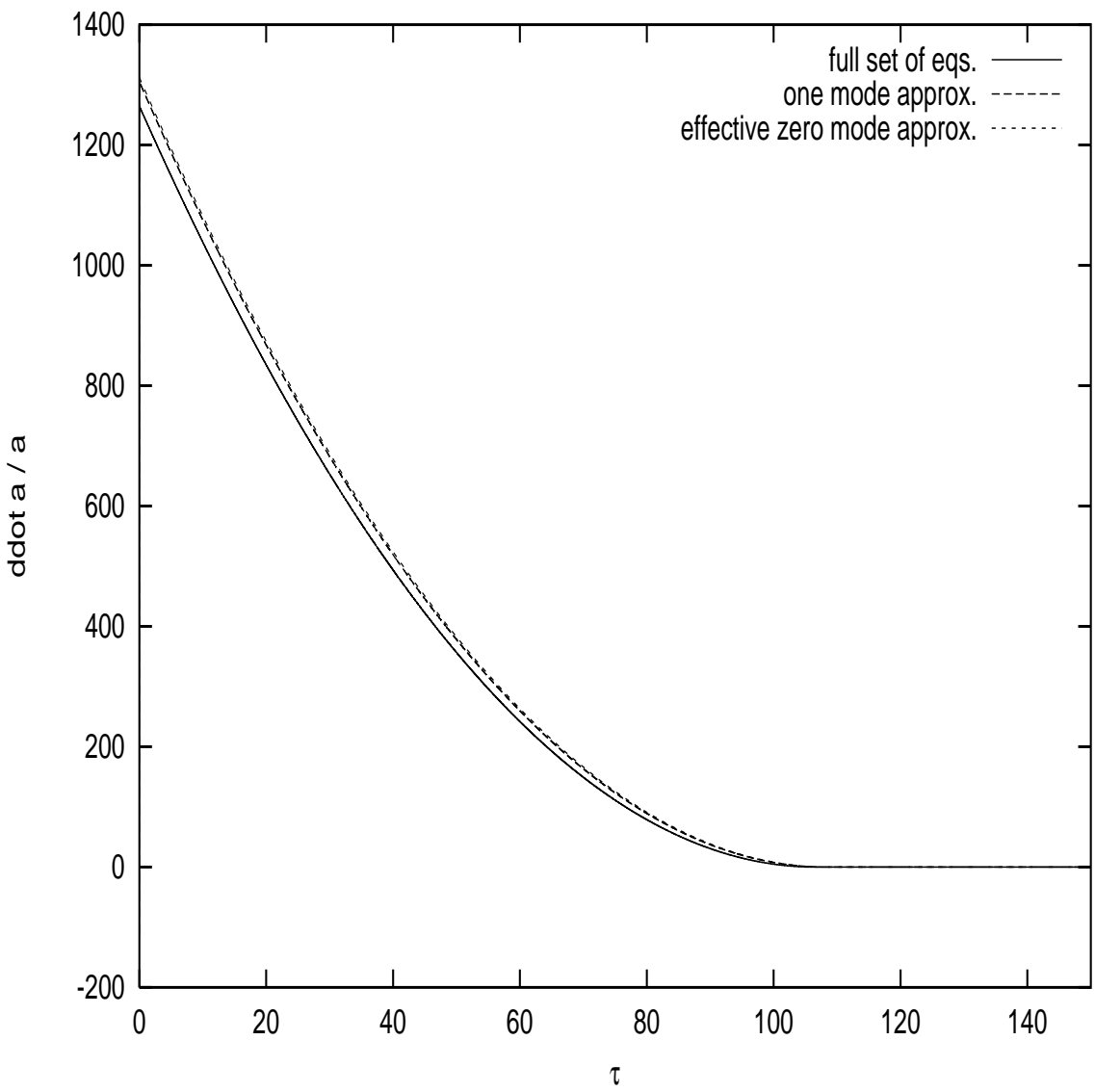


Figure 4: Tsunami inflation: $\frac{\ddot{a}(\tau)}{a(\tau)}$, it shows that there is accelerated expansion (inflation) up to times $\tau \sim 109$. Same parameters and initial conditions as in fig. 2.

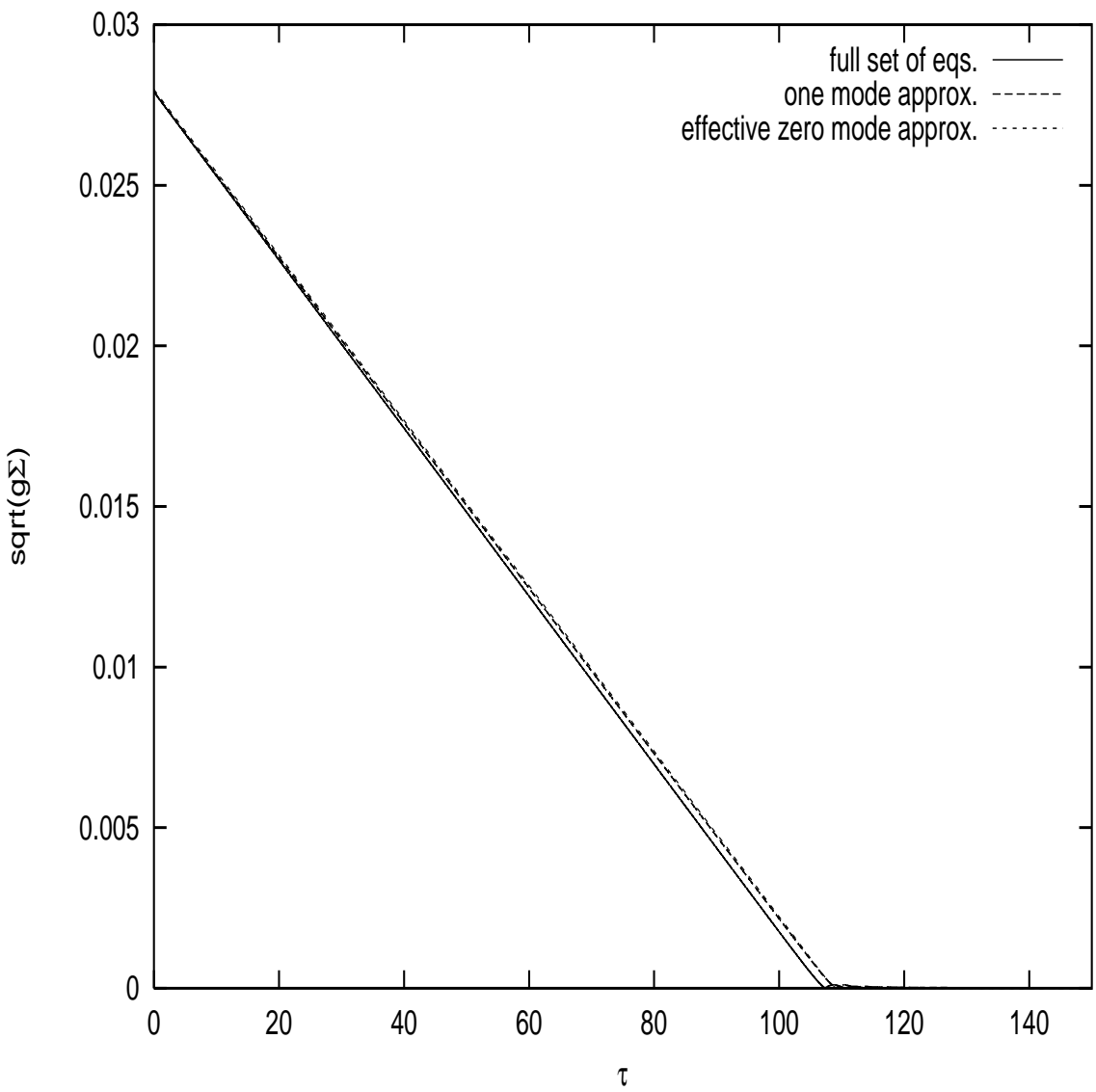


Figure 5: Tsunami inflation: $\sqrt{g\Sigma(\tau)}$, after $\tau_A \sim 0.133$, it plays the role of an effective classical field. Same parameters and initial conditions as in fig. 2.

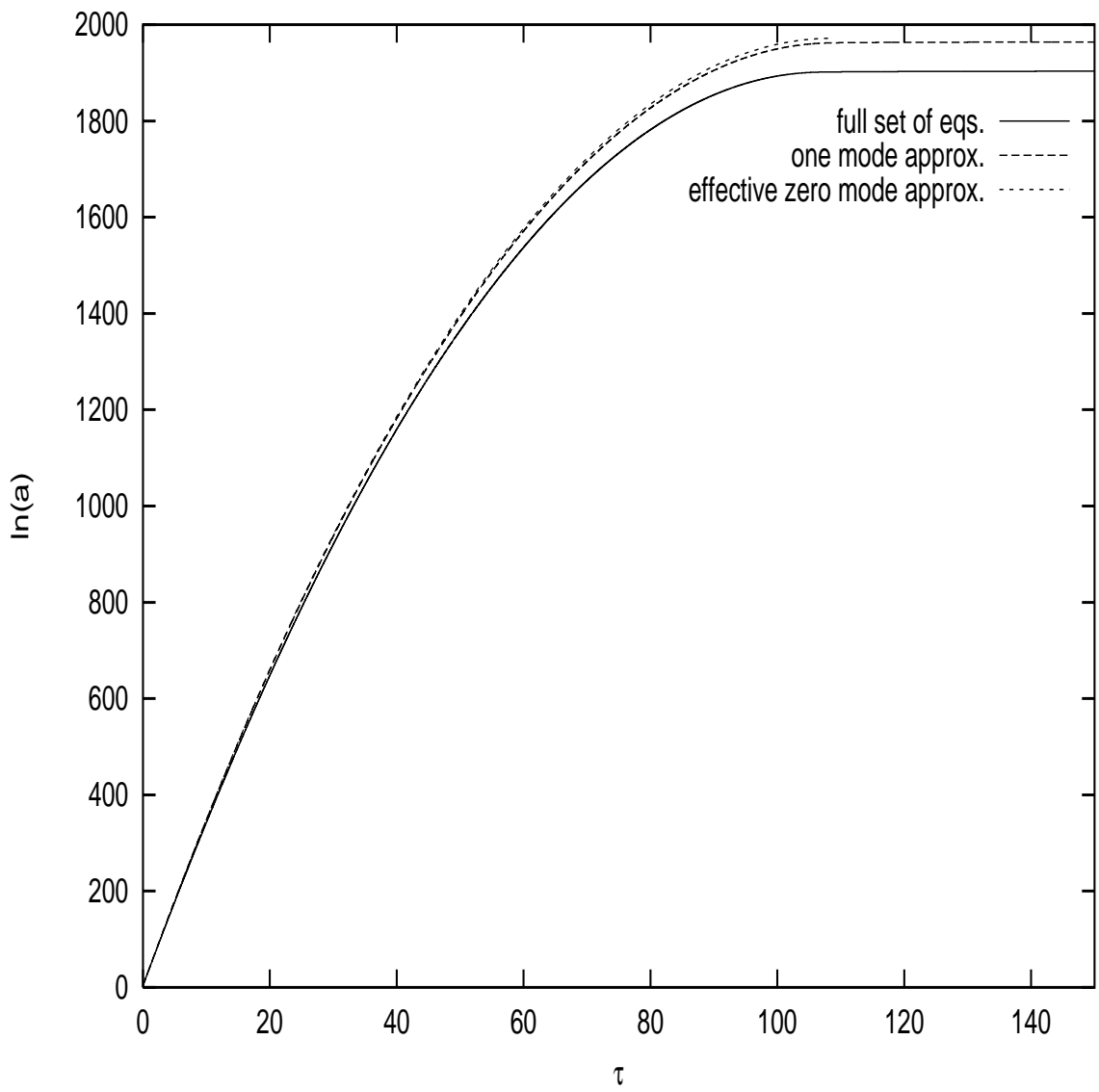


Figure 6: $\ln[a(\tau)]$ vs. τ . Same parameters and initial conditions as in fig. 2.

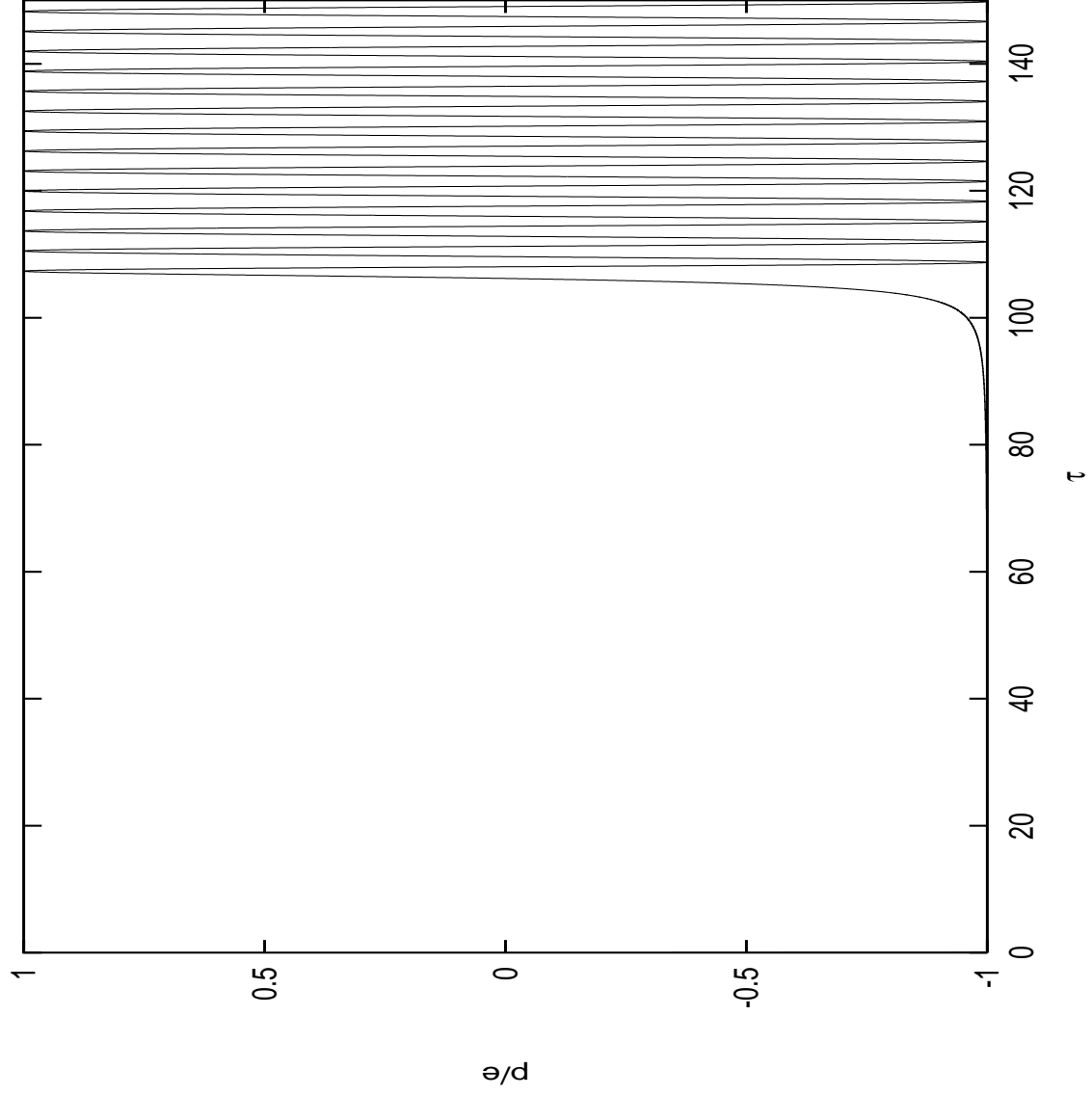


Figure 7: Tsunami inflation: $p(\tau)/\epsilon(\tau)$. It shows the onset of a matter dominated epoch after the quasi-De Sitter stage. Same parameters and initial conditions as in fig. 2.